TRANSFORMS RELATED TO THE HILBERT TRANSFORM

by

Nazim Husain Zaidi

Thesis presented for the Degree of Doctor of Philosophy
of the University of Edinburgh in the Faculty of Science
May 1964.

EX BIBL UNIV. EDINBURGH

ACKNOWLEDGEMENTS

I am extremely grateful to Dr. J. Cossar for everything connected with this thesis. He suggested the problems, and helped at each and every stage of the work. Without his help and encouragement it would have been very difficult for me to complete this work.

I should also like to thank the University of Karachi, Pakistan, for the grant of the study leave which has enabled me to come over to Edinburgh to undertake this research.

C O N T E N T S

Introduction.....1

CHAPTER I

A series inversion formula

(1.1). Titchmarsh [2] Theorem 1.....8
(1.2). A theorem on inversion.....9
(1.3). Proofs of the lemmas.....10
(1.4). Proof of Theorem (1.2).....17
(1.5). A remark.....22
(1.6). Extension of Theorem (C).....23

CHAPTER II

The Goldberg transform

(2.1). Definition.....25
(2.2). The theorem $f^* = \phi'$.....25
(2.3). The converse of Theorem (2.2).....28
(2.4). The theorem $f^* = f'$.....29
(2.5). Alternative proof of Theorem (H).....31
(2.6). The theorem $f^* = f' + f'.o.\bar{\sigma} - f''.o.\sigma$.....34
(2.7). The theorem $(f^*)' = (f')^*$.....35
(2.8). The theorem $f' = -(g^*)' - \sum_{\kappa} g^*.o.\delta_{\kappa}^*$.....35
(2.9). A remark.....36

CHAPTER III

The Boas transform

(3.1). Definition.....37
(3.2). The Boas transform with $h(u) = 0$.....39
(3.3). The third proof of Theorem (H).....43
(3.4). An inversion theorem.....45
(3.5). Another inversion theorem.....50
(3.6). Extension of Theorem (I).....51
(3.7). Extension of Theorem (J).....51
(3.8). Extension of Theorem (E).....52
(3.9). Extension of Theorem (G).....53

CHAPTER IV

The Compound Boas transforms

(4.1). Definitions.....55
(4.2). A theorem on generating function and kernel.....56
(4.3). An equivalence theorem.....58
(4.4). Another equivalence theorem.....60
(4.5). A general theorem on g.f. and kernel.....70

CHAPTER V

The differential operator h(D) and the Boas transform

(5.1). Extension of Theorem (M).....71
(5.2). Extension of Theorem (2.2).....72
(5.3). The function sinx/x.....77
(5.4). Inversion formula for f_3^*.....83
(5.5). Inversion formula for f_h^* (h:odd polynomial).....85

CHAPTER VI

Even generating function

(6.1). The Boas transform f_ν^* (ν even).....88
(6.2). The theorem $f_h^* = h(iD)\phi_1 + \phi_2$.....92
(6.3). Inversion formula for f_2^*.....93
(6.4). The Boas transform f_h^* (h:even function).....98
(6.5). Extensions of Theorems (3.7) & (4.3).....99
(6.6). Inversion formula for f_h^* ($h \neq 0$).....101
(6.7). Inversion and the operator $1/h(iD)$.....103
(6.8). Inversion formula when $h(u) = u\,p(u)$.....111

CHAPTER VII

Some other results

(7.1). Generalisation of Theorem (E).....114
(7.2). An extension of Theorem (7.1).....116
(7.3). Theorems (7.1) &(7.2) and f_h^*.....117
(7.4). Another form of the Boas transform.....118
(7.5). Some properties of the transform(7.4.2).....120
(7.6). A generalisation of the Boas transform.....124

CHAPTER VIII

Almost periodic functions
and the Goldberg transform

(8.1). Definitions.....125
(8.2). A theorem on $(R, \lambda, 1)$ summability of the conjugate series.....128
(8.3). Another theorem on summability.....135
(8.4). A theorem on the convergence of the conjugate series.....138
(8.5). Another theorem on convergence.....140

References.....146

Appendix.....147

بسم الله الرحمن الرحيم

INTRODUCTION

The transform

$$g(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(t)}{t-x}\, dt \qquad \ldots.(0.1)$$

where 'P' stands for the Cauchy principal value at t = x, and known as the Hilbert transform, has been the topic of many a research work since the beginning of this century. Many properties of 'g(x)' were thus discovered. Some of these, relevant to our work here, are as follows :-

THEOREM (A). If f(x) belongs to L^2 (*), then g(x) exists almost everywhere (a.e) and belongs to L^2, and reciprocally

$$-f(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{g(t)}{t-x}\, dt. \qquad \ldots.(0.2)$$

If F(x) be the Fourier transform of f(x), defined as

$$F(x) = \underset{a\to\infty}{l.i.m} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(t)\, e^{ixt}\, dt$$

where l.i.m stands for limit in the mean square, and G(x) be the Fourier transform of g(x), then

$$G(x) = -i\, \mathrm{sgn}x\, F(x). \qquad \ldots.(0.3)$$

It could be seen that f(x) can reciprocally be expressed in terms of F(x) as

$$f(x) = \underset{a\to\infty}{l.i.m} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} F(t)\, e^{-ixt}\, dt.$$

(*) L^2 means $L^2(-\infty, \infty)$. However if $\{f(x)\}^2$ is integrable in (a,b) we may say that $f(x) \in L^2(a,b)$.

We may call f(x) the inverse Fourier transform of F(x).

THEOREM (B). [Parseval's Formula]. If f(x) and $\phi(x)$ be any two functions of L^2 and g(x) and $\psi(x)$ be their respective Hilbert transforms, then

$$\int_{-\infty}^{\infty} f(x)\,\psi(x)\,dx = -\int_{-\infty}^{\infty} g(x)\,\phi(x)\,dx .$$

In the domain of the infinite series there is an analogue of the Hilbert transform, namely

$$b_m = \frac{1}{\pi} \sum_{-\infty}^{\infty} \frac{a_n}{n+m+\frac{1}{2}} \qquad \ldots\ldots(0.4)$$

with the reciprocal relation

$$a_m = \frac{1}{\pi} \sum_{-\infty}^{\infty} \frac{b_n}{n+m+\frac{1}{2}} . \qquad \ldots\ldots(0.5)$$

Here also, as in Theorem (A)

$$\sum_{-\infty}^{\infty} |b_n|^2$$

is convergent, (in Theorem (A), $g \in L^2$), and the reciprocal relation (0.5) holds if

$$\sum_{-\infty}^{\infty} |a_n|^2$$

is convergent. This result, and as a matter of fact both Theorems (A) & (B), hold for a general index $p > 1$. These were proved, among others, by Titchmarsh [1]. He has also proved, in this connection, a theorem namely

THEOREM (C). [(A), - p.321] Let $\{a_n\}$ denote any sequence of numbers such that the series

$$\sum_{-\infty}^{\infty} |a_n|^p \qquad (p > 1)$$

is convergent, and let b_m be as given by (0.4) for all values of m. Then the series

$$\sum_{n=-\infty}^{\infty} |b_n|^p$$

is also convergent, and there is a number N_p depending only on p such that

$$\sum_{n=-\infty}^{\infty} |b_n|^p \leq N_p^p \sum_{n=-\infty}^{\infty} |a_n|^p.$$

In yet another paper [2] Titchmarsh has discussed general conditions for the formulae (0.4) and (0.5) without taking into consideration the convergence of

$$\sum_{n=-\infty}^{\infty} |a_n|^p. \qquad (p \geq 1)$$

In Chapter 1, we discuss these conditions and prove some results rather more general than those of Titchmarsh [2].

Apart from the above analogue of the Hilbert transform there is an integral transform, namely

$$f^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(t) \frac{\sin(t-x)}{(t-x)^2} dt \qquad \ldots\ldots(0.6)$$

which is very closely related to the Hilbert transform. This transform was first discussed by Boas [4] both in the above form and in a much more general form, viz

$$f^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(t) \frac{H(t-x)}{(t-x)} dt \qquad \ldots\ldots(0.7)$$

where $$H(t) = \int_{-1}^{1} \cos st \, d\omega(s).$$

with $\omega(x)$ as a continuous function of bounded variation on $(-1,1)$. Boas also proved a theorem in this connection, namely

THEOREM (D). [(2), - p.295]. Let $f(x)$ belong to L^2; let $F(x)$ be the Fourier transform of $f(x)$ and let $H(0) = 1$. A necessary and sufficient condition that $F(x) = 0$ (a.e) on $(-1,1)$ is that

$$f^{**}(x) = -f(x) \quad \text{(a.e)}. \qquad \ldots\ldots(0.8)$$

Where $f^{*}(x)$ is defined by (0.7) and $f^{**}(x)$ is related to $f^{*}(x)$ as $f^{*}(x)$ is to $f(x)$. Under the circumstances an obvious corollary to (0.8) is by (0.2)

$$f^{*}(x) = g(x) \quad \text{(a.e)}. \qquad \ldots\ldots(0.9)$$

Goldberg [5] discussed (0.6) in greater detail, and established the following theorems :-

THEOREM (E). [(1), - p.201]. If $f(x)$ belongs to L^2, $g(x)$ be its Hilbert transform and $f^{*}(x)$ be its (0.6) transform, then

$$f^{*} = g - g.o.\,\delta$$

where
$$\delta(x) = \sqrt{\frac{2}{\pi}}\ \frac{1-\cos x}{x^2} \qquad \ldots\ldots(0.10)$$

and the convolution $g.o.\delta$ is defined as

$$g.o.\delta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t)\,\delta(t)\,dt.$$

THEOREM (F). [(2), - p.203]. If f, g and f^{*} be as in Theorem (E), then

$$g = f^{*} + \sum_{K=1}^{\infty} f^{*}.o.\,\delta_K$$

the sum converging in the L^2 mean, and

$$\left.\begin{aligned} \delta_1 &= \delta \\ \delta_k &= \delta_{k-1} \circ \delta \qquad (k = 2,3,\ldots). \end{aligned}\right\} \quad \ldots\ldots(0.11)$$

THEOREM (G). [(3), - p.203]. If $f(x)$ belongs to L^2 and $f^{**}(x)$ be the (0.6) tansform of $f^*(x)$, then

$$f^{**} = -f + 2\delta \circ f - \delta \circ \delta \circ f .$$

Recently Heywood [6] established the following results including an inversion formula for the transform (0.6).

THEOREM (H). [(1), - p.163]. If f and f^* be as above, then

$$-f(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f^*(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 \frac{\sin ut}{u}\, du \right\} dt \ldots\ldots(0.12)$$

THEOREM (I). [(2), - p.163].

$$\lim_{\substack{\xi \to \infty \\ \eta \to \infty}} \int_{-\eta}^{\xi} f^*(x)\, dx = 0.$$

THEOREM (J). [(3), - p.167]. (Analogue of Theorem (B)). If $\phi(x)$ also belongs to L^2, with $\phi^*(x)$ as its (0.6) transform, then

$$\int_{-\infty}^{\infty} \phi(x)\, f^*(x)\, dx = -\int_{-\infty}^{\infty} f(x)\, \phi^*(x)\, dx .$$

For convenience we shall call the (0.6) transform the 'Goldberg' transform and (0.7) the 'Boas' transform. In Chapter 2 we make a

further study of the Goldberg transform, and prove our main result, namely that the class of the Goldberg transforms $\phi(x)$ of functions of L^2 is the subclass of L^2 for which

$$\int_x^\infty \phi(t)\, dt \quad \text{also belongs to } L^2.$$

Chapters 3 - 6 are devoted to the study of the general Boas transform. Results similar to those of Goldberg, Heywood, and our own of Chapter 2 are obtained in the general case also. It is also shown there that the Hilbert transform itself is a particular case of the Boas transform. Chapter 7 contains a few results which could not find a place in the preceding chapters. It also contains a brief description of a different form of the Boas transform, and as the theory here presents no new features, we simply state the corresponding results without formally giving their proofs.

Lastly in Chapter 8, an application of the theory of the Goldberg transform is made to the Fourier series of the Bohr almost periodic (U.a.p) functions. The main result, there, is that under suitable conditions the conjugate series of the Fourier series of a (U.a.p) function may be made to be summable $(R, \lambda, 1)$ to the Goldberg transform of the function concerned.

Throughout this thesis we have had the occasion of using several times some of the well known properties of the L^2 functions. These, as ststed in Titchmarsh [8], are as follows :-

THEOREM (K). [(49), - p.70]. If f, F and g, G are Fourier transforms of the L^2 class, then

$$\int_{-\infty}^{\infty} F(x)\, G(x)\, dx \quad = \quad \int_{-\infty}^{\infty} f(x)\, g(-x)\, dx\,.$$

THEOREM (L). [(64), - p.90]. If f, F and g, G be as above, then

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(u)\, f(x-u)\, du \qquad \text{and} \qquad F(x)G(x)$$

are Fourier transforms.

THEOREM (M). [(68), - p.92]. If f(x) be absolutely continuous and f(x) and f'(x) belong to L^2, then both F(x) the Fourier transform of f(x) and xF(x) belong to L^2 and conversely.

In fact, here, we observe that

$$f(x) \longrightarrow 0 \quad \text{as } x \longrightarrow \pm \infty, \qquad \ldots.(0.13)$$

and if the Fourier transform of f'(x) be $\Phi(x)$, then

$$\Phi(x) \quad = \quad -ix\, F(x). \qquad \ldots.(0.14)$$

Theorems (A) & (B) may also be referred to Titchmarsh [8] as [(91), - p.122], and [(102), - p.138] respectively, and relation (0.3) may be found on page 121.

The above results will now be referred to, in the main body of our thesis, as Theorems (A), (B), etc. and not , in general, by their original names and titles.

At the end of the thesis we also provide a list of some L^2 functions with their Fourier, Hilbert, and Goldberg transforms, in the form of an appendix. These functions occur repeatedly in our discussions.

CHAPTER I

A SERIES INVERSION FORMULA

(1.1) Titchmarsh [2] showed that if the series

$$\frac{1}{\pi}\sum_{-\infty}^{\infty}\frac{a_n}{n+m+\frac{1}{2}} \qquad \ldots\ldots(1.1.1)$$

is convergent in the ordinary sense to, say, b_m, then reciprocally

$$a_m = \lim_{N\to\infty}\frac{1}{\pi}\sum_{-N}^{N}\left(1-\frac{|n|}{N}\right)\frac{b_n}{n+m+\frac{1}{2}}. \qquad \ldots\ldots(1.1.2)$$

Where by convergence in the ordinary sense of the series of the form

$$\sum_{-\infty}^{\infty} u_n \qquad \ldots\ldots(1.1.3)$$

is meant that the two series

$$\sum_{1}^{\infty} u_n \qquad \text{and} \qquad \sum_{1}^{\infty} u_{-n}$$

are both convergent. The series (1.1.3) is convergent in the restricted sense if the series

$$\sum_{1}^{\infty}\left(u_n + u_{-n}\right)$$

is convergent. Similar definitions hold for summability in the two senses. Titchmarsh [2] for his above theorem took the ordinary convergence of (1.1.1) and dismissed its restricted convergence on the basis of the following example.

EXAMPLE Let $a_n = 1$ for all n, then the series (1.1.1) converges in the restricted sense to the sum zero, ie

$$b_m = 0 \qquad \text{for all } m.$$

And so (1.1.2) is not satisfied.

Nevertheless we show here that if the sequence $\{a_n\}$ be subjected to some suitable conditions then even the restricted convergence of (1.1.1) could give the (C,1) summability, in the restricted sense, of

$$\sum_{-\infty}^{\infty} \frac{b_n}{n+m+\frac{1}{2}}.$$

The suitable conditions referred to above are, in fact, similar to those of Cossar [3] Theorem 1, where he proves a similar result for integrals. Our theorem, which is a series version of Cossar's theorem, is as follows :-

(1.2) THEOREM : If the sequence $\{a_n\}$ be such that

$$\sum_{1}^{\infty} \frac{a_n - a_{-n}}{n} \qquad \ldots\ldots(1.2.1)$$

is convergent, and $$\sum_{N}^{\infty} \frac{a_n + a_{-n}}{n^2} = o\left(\frac{1}{N}\right) \quad (N \to \infty) \qquad \ldots\ldots(1.2.2)$$

then the series (1.1.1) converges in the restricted sense to, say, b_m and reciprocally (1.1.2) holds.

For the proof of the above theorem we require the following lemmas;

LEMMA 1

$$\text{If} \quad F_M(r) = \frac{1}{M}\left\{\sum_{m=1}^{M} \frac{1}{m+r+\frac{1}{2}} + \sum_{m=0}^{M-1} \frac{1}{m-r+\frac{1}{2}}\right\}$$

$$\text{then} \quad \sum_{r=0}^{\infty} \left|F_M(r)\right| \quad \text{is bounded for all M.}$$

LEMMA 2

$$\text{If} \quad G_M(r) = \frac{1}{M+r+3/2} + \frac{1}{M-r+\frac{1}{2}}$$

$$\text{then} \quad \lim_{M\to\infty} \frac{1}{M} \sum_{r=0}^{\infty} \left| G_M(r) \right| = 0.$$

LEMMA 3

$$\text{If} \quad P_M(r) = \left\{ \sum_{m=0}^{M} + \sum_{m=-1}^{-M-1} + \frac{r}{M} \left(\sum_{0}^{M} - \sum_{-1}^{-M-1} \right) \right\} \frac{1}{m+r+\frac{1}{2}}$$

$$\text{then} \quad \sum_{r=0}^{\infty} \left| P_M(r+1) - P_M(r) \right| \quad \text{is bounded for all M.}$$

LEMMA 4

$$\text{If} \quad Q_M(r) = \sum_{m=0}^{M} \left\{ \frac{1}{m+r+\frac{1}{2}} - \frac{1}{m-r+\frac{1}{2}} \right\}$$

$$\text{then} \quad \lim_{M\to\infty} \sum_{r=1}^{\infty} \frac{\left| Q_M(r) \right|}{r} \quad \text{exists.}$$

LEMMA 5

$$\text{If} \quad I_M(r) = \sum_{0}^{M} \frac{1}{m+r+\frac{1}{2}} - \sum_{-M-1}^{-1} \frac{1}{m+r+\frac{1}{2}}$$

$$\text{then} \quad \lim_{M\to\infty} \frac{1}{M} \sum_{r=0}^{\infty} \left| I_M(r+1) - I_M(r) \right| = 0.$$

(1.3) PROOFS OF THE LEMMAS:-

(1).

$$\frac{1}{r-m-\frac{1}{2}} > \frac{1}{r+m+\frac{1}{2}} \qquad \text{for } r \geqslant m+1.$$

$$\text{Hence} \quad \sum_{m=0}^{M-1} \frac{1}{r-m-\frac{1}{2}} > \sum_{m=1}^{M} \frac{1}{r+m+\frac{1}{2}} \qquad \text{if } r \geqslant M,$$

and so, if $N > M$

$$\sum_{r=0}^{N} \left| \sum_{m=1}^{M} \frac{1}{m+r+\frac{1}{2}} + \sum_{m=0}^{M-1} \frac{1}{m-r+\frac{1}{2}} \right|$$

$$= \sum_{r=0}^{M-1} \left| \sum_{m=1}^{M} \frac{1}{m+r+\frac{1}{2}} + \sum_{m=0}^{M-1} \frac{1}{m-r+\frac{1}{2}} \right| + \sum_{r=M}^{N} \left\{ \sum_{m=0}^{M-1} \frac{1}{r-m-\frac{1}{2}} - \sum_{m=1}^{M} \frac{1}{m+r+\frac{1}{2}} \right\}$$

$$\leq \sum_{r=0}^{M-1} \sum_{m=1}^{M} \frac{1}{m+r+\frac{1}{2}} + \sum_{r=0}^{M-1} \left| \sum_{m=0}^{M-1} \frac{1}{m-r+\frac{1}{2}} \right| + \sum_{r=M}^{N} \sum_{m=0}^{M-1} \frac{1}{r-m-\frac{1}{2}} - \sum_{r=M}^{N} \sum_{m=0}^{M} \frac{1}{m+r+\frac{1}{2}}$$

$$= I_1 + I_2 + I_3 - I_4 \qquad \text{(say).}$$

Now $$I_1 = \sum_{r=1}^{M} \sum_{m=1}^{M} \frac{1}{m+r-\frac{1}{2}}$$ (by setting r - 1 for r).

$$I_2 = \sum_{m=1}^{M} \frac{1}{m-\frac{1}{2}} + \sum_{r=1}^{M-1} \left| \sum_{m=0}^{M-1} \frac{1}{m-r+\frac{1}{2}} \right|.$$

$$I_3 = \left(\sum_{r=M}^{2M} + \sum_{2M+1}^{N} \right) \sum_{m=0}^{M-1} \frac{1}{r-m-\frac{1}{2}},$$

and $$\sum_{r=2M+1}^{N} \sum_{m=0}^{M-1} \frac{1}{r-m-\frac{1}{2}} = \sum_{r=M}^{N-M-1} \sum_{m=1}^{M} \frac{1}{m+r+\frac{1}{2}}$$

(by setting M + r + 1 for r, and M - m for m.)

Hence $$I_3 - I_4 = \sum_{r=M}^{2M} \sum_{m=0}^{M-1} \frac{1}{r-m-\frac{1}{2}} - \sum_{r=N-M}^{N} \sum_{m=1}^{M} \frac{1}{m+r+\frac{1}{2}}$$

$$= \sum_{r=0}^{M} \sum_{m=1}^{M} \frac{1}{m+r+\frac{1}{2}} - \sum_{r=N-M}^{N} \sum_{m=1}^{M} \frac{1}{m+r+\frac{1}{2}}$$

(by setting M + r for r, and M - m for m in the first term.)

Thus $$I_1 + I_3 - I_4 = 2 \sum_{r=0}^{M} \sum_{m=1}^{M} \frac{1}{m+r-\frac{1}{2}} - \sum_{m=1}^{M} \frac{1}{m-\frac{1}{2}} - \sum_{r=N-M}^{N} \sum_{m=1}^{M} \frac{1}{m+r+\frac{1}{2}}.$$

Now $$\lim_{N \to \infty} \sum_{r=N-M}^{N} \sum_{m=1}^{M} \frac{1}{m+r+\frac{1}{2}} = 0. \qquad \ldots\ldots(1.3.1)$$

Also $$\sum_{r=0}^{M} \sum_{m=1}^{M} \frac{1}{m+r-\frac{1}{2}} = \sum_{m=1}^{M} m \left\{ \frac{1}{m-\frac{1}{2}} + \frac{1}{2M-m+\frac{1}{2}} \right\}$$

$$= 4M \sum_{1}^{M} \frac{m}{2m-1} \cdot \frac{1}{2M-m+\frac{1}{2}}$$

$$< 4M \sum_{1}^{M} \frac{1}{2M-m+\frac{1}{2}}$$

$$< 4M. \qquad \ldots\ldots(1.3.2)$$

As $$\left| \sum_{m=0}^{M-1} \frac{1}{m-r+\frac{1}{2}} \right|_{r=\nu} = \left| \sum_{m=0}^{M-1} \frac{1}{m-r+\frac{1}{2}} \right|_{r=M-\nu}$$

$$= \sum_{m=\nu}^{M-\nu-1} \frac{1}{m+\frac{1}{2}} \qquad \text{for } \nu < \frac{M}{2}$$

$$= 0 \qquad \text{if } M \text{ is even and } \nu = \frac{M}{2},$$

we have, for even M,

$$\sum_{r=1}^{M-1} \left| \sum_{m=0}^{M-1} \frac{1}{m-r+\frac{1}{2}} \right| = 2 \left\{ \sum_{m=1}^{M-2} + \sum_{m=2}^{M-3} + \cdots\cdots + \sum_{m=\frac{M}{2}-1}^{\frac{M}{2}} \right\} \frac{1}{m+\frac{1}{2}}$$

$$= 2 \left\{ \left(\frac{1}{3/2} + \frac{1}{M-3/2} \right) + 2 \left(\frac{1}{5/2} + \frac{1}{M-5/2} \right) + \cdots + \frac{M-2}{2} \left(\frac{1}{\frac{M-1}{2}} + \frac{1}{\frac{M+1}{2}} \right) \right\}$$

$$= 2M \left\{ \frac{2}{3} \frac{1}{M-3/2} + \frac{4}{5} \frac{1}{M-5/2} + \cdots\cdots + \frac{M-2}{M-1} \frac{1}{\frac{M+1}{2}} \right\}$$

$$< \frac{2M(M-2)}{M}$$

$$< 2(M-2) \qquad (\text{for } M > 2), \qquad \ldots\ldots(1.3.3)$$

and, for odd M (>1),

$$\sum_{r=1}^{M-1}\left|\sum_{m=0}^{M-1}\frac{1}{m-r+\frac{1}{2}}\right| = \sum_{r=1}^{M-2}\left|\sum_{m=0}^{M-2}\frac{1}{m-r+\frac{1}{2}}\right| + \sum_{r=1}^{M-2}\frac{1}{M-r+\frac{1}{2}} + \frac{1}{1/2} + \sum_{m=0}^{M-2}\frac{1}{M-m-3/2}$$

$$< 2(M-3) + \frac{2}{3}(M-2) + 2 + 2(M-1)$$

(by applying (1.3.3) to the first term)

$$< \frac{14}{3}(M-1). \qquad \ldots\ldots(1.3.4)$$

Hence it follows from (1.3.1), (1.3.2), (1.3.3) and (1.3.4) that

$$\sum_{r=0}^{\infty}\left|F_M(r)\right| \qquad \text{is bounded for all } M.$$

This completes the proof of Lemma 1.

(2). Now

$$\sum_{r=0}^{N}\left|G_M(r)\right| = \sum_{r=0}^{M}\left\{\frac{1}{M+r+3/2}+\frac{1}{M-r+\frac{1}{2}}\right\} + \sum_{r=M+1}^{N}\left\{\frac{1}{r-M-\frac{1}{2}}-\frac{1}{r+M+3/2}\right\}$$

$$= \sum_{r=0}^{M}\left\{\frac{1}{M+r+3/2}+\frac{1}{M-r+\frac{1}{2}}\right\} + \left\{\sum_{r=M+1}^{3M+2}+\sum_{3M+3}^{N}\right\}\frac{1}{r-M-\frac{1}{2}} - \sum_{M+1}^{N}\frac{1}{M+r+3/2}$$

$$= \sum_{r=0}^{2M+1}\frac{1}{r+\frac{1}{2}} + \sum_{r=M+1}^{3M+2}\frac{1}{r-M-\frac{1}{2}} + \sum_{M+1}^{N-2M-2}\frac{1}{M+r+3/2} - \sum_{M+1}^{N}\frac{1}{M+r+3/2}$$

$$= 2\sum_{0}^{2M+1}\frac{1}{r+\frac{1}{2}} - \sum_{N-2M-1}^{N}\frac{1}{M+r+3/2}$$

$$= 2\sum_{0}^{2M+1}\frac{1}{r+\frac{1}{2}} + o(1).$$

Hence $$\sum_{r=0}^{\infty}\left|G_M(r)\right| = 2\sum_{0}^{2M+1}\frac{1}{r+\frac{1}{2}},$$

and $$\lim_{M\to\infty}\frac{1}{M}\sum_{0}^{\infty}\left|G_M(r)\right| = \lim_{M\to\infty}\frac{2}{M}\sum_{0}^{2M+1}\frac{1}{r+\frac{1}{2}} = 0.$$

Thus we get Lemma 2.

(3). $$P_M(r) = \sum_0^M \frac{1}{m+r+\frac{1}{2}} - \sum_0^M \frac{1}{m-r+\frac{1}{2}} + \frac{r}{M}\left(\sum_0^M \frac{1}{m+r+\frac{1}{2}} + \sum_0^M \frac{1}{m-r+\frac{1}{2}}\right)$$

$$P_M(r+1) - P_M(r) = \frac{1}{M+r+3/2} - \frac{1}{r+\frac{1}{2}} + \frac{1}{r+\frac{1}{2}} + \frac{1}{M-r+\frac{1}{2}} +$$

$$+ \frac{r}{M}\left\{\frac{1}{M+r+3/2} - \frac{2}{r+\frac{1}{2}} - \frac{1}{M-r+\frac{1}{2}}\right\} +$$

$$+ \frac{1}{M}\left\{\sum_1^{M+1} \frac{1}{m+r+\frac{1}{2}} + \sum_{-1}^{M-1} \frac{1}{m-r+\frac{1}{2}}\right\}$$

$$= \frac{1}{M}\left\{\sum_1^M \frac{1}{m+r+\frac{1}{2}} + \sum_0^{M-1} \frac{1}{m-r+\frac{1}{2}} - \frac{1/2}{M+r+3/2} - \frac{1/2}{M-r+\frac{1}{2}}\right\},$$

and so following the notations of Lemmas 1, & 2,

$$\sum_{r=0}^{\infty} \left| P_M(r+1) - P_M(r) \right| \leqslant \sum_{r=0}^{\infty} \left| F_M(r) \right| + \frac{1}{2M} \left| G_M(r) \right|$$

$$\leqslant K \qquad \text{(by the same two Lemmas)}$$

and this gives Lemma 3.

(4). We first show that $Q_M(r) > 0$, for all positive integral values of r. That it is positive for $r > M$ is obvious. When $r \leqslant M$, we consider separately the two cases of

(i) $2r \leqslant M$

(ii) $2r > M$

(i) $2r \leqslant M$

$$\sum_{m=0}^{M} \frac{1}{m-r+\frac{1}{2}} = \sum_{2r}^{M} \frac{1}{m-r+\frac{1}{2}} = \sum_0^{M-2r} \frac{1}{m+r+\frac{1}{2}} < \sum_0^M \frac{1}{m+r+\frac{1}{2}}.$$

(ii) (a) $2r = M + 1$

(b) $2r > M + 1$

$$\sum_{m=0}^{M} \frac{1}{m-r+\frac{1}{2}} = \sum_{0}^{2r-M-2} \frac{1}{m-r+\frac{1}{2}} < 0.$$

Thus $Q_M(r) > 0$ in all the above cases.

Hence $$\sum_{r=1}^{\infty} \frac{|Q_M(r)|}{r} = \sum_{r=1}^{\infty} \frac{Q_M(r)}{r},$$

and as $$\frac{Q_M(r)}{r} = \sum_{0}^{M} \left\{ \frac{1}{r(m+r+\frac{1}{2})} - \frac{1}{r(m-r+\frac{1}{2})} \right\}$$

$$= -\sum_{0}^{M} \frac{1}{m+\frac{1}{2}} \left\{ \frac{1}{m+r+\frac{1}{2}} + \frac{1}{m-r+\frac{1}{2}} \right\}$$

$$\sum_{r=1}^{N} \frac{Q_M(r)}{r} = -\sum_{m=0}^{M} \frac{1}{m+\frac{1}{2}} \left[\sum_{r=1}^{N} \left\{ \frac{1}{m+r+\frac{1}{2}} + \frac{1}{m-r+\frac{1}{2}} \right\} \right], \quad (r \neq 0)$$

and $$\sum_{r=1}^{N} \frac{1}{m-r+\frac{1}{2}} = \sum_{r=2m+1}^{N} \frac{1}{m-r+\frac{1}{2}} = -\frac{1}{m+\frac{1}{2}} - \sum_{r=1}^{N-2m-1} \frac{1}{m+r+\frac{1}{2}}.$$

Hence $$\sum_{r=1}^{N} \frac{Q_M(r)}{r} = \sum_{m=0}^{M} \frac{1}{(m+\frac{1}{2})^2} - \sum_{m=0}^{M} \frac{1}{m+\frac{1}{2}} \sum_{r=N-2m}^{N} \frac{1}{m+r+\frac{1}{2}},$$

and as $\lim_{N\to\infty} \sum_{r=N-2m}^{N} \frac{1}{m+r+\frac{1}{2}} = 0$, $\sum_{r=1}^{\infty} \frac{Q_M(r)}{r} = \sum_{m=0}^{M} \frac{1}{(m+\frac{1}{2})^2}$.

Since $$\sum_{m=0}^{\infty} \frac{1}{(m+\frac{1}{2})^2} = \frac{\pi^2}{2}$$

$$\sum_{r=1}^{\infty} \frac{Q_M(r)}{r} = \frac{\pi^2}{2} + o(1),$$

and so we get Lemma 4.

(5). $$I_M(r+1) - I_M(r) = \frac{1}{M+r+3/2} - \frac{2}{r+\frac{1}{2}} - \frac{1}{M-r+\frac{1}{2}}$$

$$= \frac{1}{M+r+3/2} + \frac{1}{M-r+\frac{1}{2}} - 2\left\{\frac{1}{r+\frac{1}{2}} + \frac{1}{M-r+\frac{1}{2}}\right\}.$$

Thus $$\left| I_M(r+1) - I_M(r)\right| \leq \left|\frac{1}{M+r+3/2} + \frac{1}{M-r+\frac{1}{2}}\right| + 2\left|\frac{1}{r+\frac{1}{2}} + \frac{1}{M-r+\frac{1}{2}}\right|,$$

and so using the notations of Lemma 2,

$$\lim_{M\to\infty} \frac{1}{M} \sum_{r=0}^{\infty} \left| I_M(r+1) - I_M(r)\right| \leq \lim_{M\to\infty} \frac{1}{M} \sum_{0}^{\infty} |G_M(r)| +$$

$$+ \lim_{M\to\infty} \frac{2}{M} \sum_{0}^{\infty} \left|\frac{1}{r+\frac{1}{2}} - \frac{1}{M+\frac{1}{2}-r}\right|$$

$$\leq \lim_{M\to\infty} \frac{2}{M} \sum_{r=0}^{\infty} \left|\frac{1}{r+\frac{1}{2}} + \frac{1}{M-r+\frac{1}{2}}\right| \text{ (by Lemma 2).}$$

Now $$\sum_{r=0}^{N} \left|\frac{1}{r+\frac{1}{2}} + \frac{1}{M-r+\frac{1}{2}}\right| = \sum_{0}^{M} \left\{\frac{1}{r+\frac{1}{2}} + \frac{1}{M-r+\frac{1}{2}}\right\} + \sum_{M+1}^{N} \left\{\frac{1}{r-M-\frac{1}{2}} - \frac{1}{r+\frac{1}{2}}\right\}$$

$$= 2\sum_{r=0}^{M} \frac{1}{r+\frac{1}{2}} + \sum_{0}^{N-M-1} \frac{1}{r+\frac{1}{2}} - \sum_{M+1}^{N} \frac{1}{r+\frac{1}{2}}$$

(by setting M + r + 1 for r in the first term of the 2nd bracket),

$$= 3\sum_{r=0}^{M} \frac{1}{r+\frac{1}{2}} - \sum_{N-M}^{N} \frac{1}{r+\frac{1}{2}}$$

The second term on the right $\to 0$, as $N \to \infty$, and

$$\lim_{M\to\infty} \frac{1}{M} \sum_{r=0}^{M} \frac{1}{r+\frac{1}{2}} = 0.$$

Hence we get Lemma 5.

(1.4) PROOF OF THEOREM (1.2)

$$\sum_{n=-N}^{N} \frac{a_n}{n+m+\frac{1}{2}} = \frac{a_0}{m+\frac{1}{2}} + \left(\sum_{1}^{N} + \sum_{-N}^{-1}\right) \frac{a_n}{n+m+\frac{1}{2}}$$

$$= \frac{a_0}{m+\frac{1}{2}} + \sum_{n=1}^{N} \frac{(m+\frac{1}{2})(a_n+a_{-n}) - n(a_n-a_{-n})}{(m+\frac{1}{2})^2 - n^2}$$

$$= \frac{a_0}{m+\frac{1}{2}} + (m+\tfrac{1}{2}) \sum_{n=1}^{N} \frac{a_n+a_{-n}}{(m+\frac{1}{2})^2-n^2} - \sum_{n=1}^{N} \frac{n(a_n-a_{-n})}{(m+\frac{1}{2})^2-n^2} .$$

By (1.2.1), (1.2.2) and Abel's test the second and the third terms tend to a limit as $N \longrightarrow \infty$. We denote the limit of the right hand side by πb_m, and this establishes the restricted convergence of (1.1.1). To prove the inversion formula (1.1.2) we first show that, if λ_m vanishes for $|m| > c$, and

$$\mu_n = \frac{1}{\pi} \sum_{m=-c}^{c} \frac{\lambda_m}{m+n+\frac{1}{2}}$$

then
$$\sum_{-\infty}^{\infty} b_m \lambda_m = \lim_{N\to\infty} \sum_{-N}^{N} a_n \mu_n . \quad \ldots\ldots(1.4.1)$$

Now
$$\sum_{-N}^{N} a_n \mu_n = \frac{1}{\pi} \sum_{n=-N}^{N} a_n \sum_{m=-c}^{c} \frac{\lambda_m}{m+n+\frac{1}{2}}$$

$$= \frac{1}{\pi} \sum_{m=-c}^{c} \lambda_m \sum_{n=-N}^{N} \frac{a_n}{n+m+\frac{1}{2}} ,$$

and so
$$\lim_{N\to\infty} \sum_{-N}^{N} a_n \mu_n = \sum_{-c}^{c} \lambda_m b_m \qquad \text{(by (1.1.1))}$$

$$= \sum_{-\infty}^{\infty} \lambda_m b_m \quad (\text{as } \lambda_m = 0 \text{ for } |m| > c).$$

Let now

$$\lambda_m = \begin{cases} \left(1 - \frac{|m|}{M}\right) \frac{1}{m+n+\frac{1}{2}}, & (-M-1 \leq m \leq M) \\ 0 & \text{elsewhere.} \end{cases}$$

Then

$$\mu_r = \frac{1}{\pi} \sum_{m=-M-1}^{M} \left(1 - \frac{|m|}{M}\right) \frac{1}{(m+n+\frac{1}{2})(m+r+\frac{1}{2})}$$

$$= \frac{1}{\pi(n-r)} \left[\sum_{-M-1}^{-1} \left(\frac{1 - \frac{r+\frac{1}{2}}{M}}{m+r+\frac{1}{2}} - \frac{1 - \frac{n+\frac{1}{2}}{M}}{m+n+\frac{1}{2}} \right) + \right.$$

$$\left. + \sum_{0}^{M} \left(\frac{1 + \frac{r+\frac{1}{2}}{M}}{m+r+\frac{1}{2}} - \frac{1 + \frac{n+\frac{1}{2}}{M}}{m+n+\frac{1}{2}} \right) \right] \quad r \neq n.$$

Otherwise, $$\mu_n = \frac{1}{\pi} \sum_{m=-M-1}^{M} \left(1 - \frac{|m|}{M}\right) \frac{1}{(m+n+\frac{1}{2})^2}.$$

Hence if r ≠ n, using the notations of Lemmas 3, & 5, we get

$$\mu_r = \frac{1}{\pi(n-r)} \left\{ P_M(r) - P_M(n) + \frac{1}{2M} \left(I_M(r) - I_M(n) \right) \right\}.$$

Now $P_M(-r) = -P_M(r)$,

$P_M(r) \longrightarrow 0$ as $M \longrightarrow \infty$ for a fixed r,

$P_M(r) \longrightarrow 0$ as $r \longrightarrow \infty$ for a fixed M,

and $I_M(-r) = I_M(r)$,

$\frac{1}{M} I_M(r) \longrightarrow 0$ as $M \longrightarrow \infty$ for a fixed r,

$I_M(r) \longrightarrow 0$ as $r \longrightarrow \infty$ for a fixed M.

Hence by (1.4.1)

$$\sum_{-M-1}^{M} \left(1 - \frac{|m|}{M}\right) \frac{b_m}{m+n+\frac{1}{2}} = \frac{1}{\pi} \lim_{T \to \infty} \left(\sum_{-T}^{-1} + \sum_{1}^{T} \right)' \frac{a_r}{n-r} \left\{ P_M(r) - P_M(n) + \frac{1}{2M} \left(I_M(r) - I_M(n) \right) \right\} +$$

$$+ \frac{1}{\pi} \frac{a_0}{n} \left\{ P_M(0) - P_M(n) + \frac{1}{2M} \left(I_M(0) - I_M(n) \right) \right\} +$$

$$+ C_M. \qquad \ldots\ldots(1.4.2)$$

Where 'prime' means that we are to exclude the term for r = n from the summation, and

$$C_M = \frac{a_n}{\pi} \sum_{m=-M-1}^{M} \left(1 - \frac{|m|}{M}\right) \frac{1}{(m+n+\frac{1}{2})^2} .$$

When $M \longrightarrow \infty$, $C_M \longrightarrow \pi a_n$, for

$$\sum_{-\infty}^{\infty} \frac{1}{(m+n+\frac{1}{2})^2} = \pi^2 .$$

Moreover the second term on the R.H.S. of (1.4.2) tends to zero as $M \longrightarrow \infty$, and as both $P_M(n)$ and $\frac{1}{2M} I_M^{(n)}$ are independent of r, and tend to zero as $M \longrightarrow \infty$, and further by (1.2.1), (1.2.2) and Abel's test (cf. p.17)

$$\lim_{T \to \infty} \left(\sum_{-T}^{-1} + \sum_{1}^{T} \right)' \frac{a_r}{n-r} \qquad (r \neq n)$$

exists, so

$$\lim_{M \to \infty} \lim_{T \to \infty} \left(\sum_{-T}^{-1} + \sum_{1}^{T} \right)' \frac{a_r}{n-r} \left\{ P_M(n) + \frac{1}{2M} I_M(n) \right\} = 0 .$$

Thus in order to get (1.1.2) we need prove only that

$$\lim_{M \to \infty} \lim_{T \to \infty} \left(\sum_{-T}^{-1} + \sum_{1}^{T} \right)' \frac{a_r}{n-r} \left\{ P_M(r) + \frac{1}{2M} I_M(r) \right\} = 0 .$$

Let $$J_M = \lim_{T \to \infty} \left(\sum_{-T}^{-1} + \sum_{1}^{T} \right)' \frac{a_r}{n-r} P_M(r)$$

$$= \lim_{T \to \infty} \sum_{1}^{T}{}' \left\{ \phi_r + \psi_r \right\} P_M(r) .$$

Where $$\phi_r = \frac{r}{n^2-r^2}\,(a_r + a_{-r}),$$

and $$\psi_r = \frac{n}{n^2-r^2}\,(a_r - a_{-r}).$$

Again from (1.2.1), (1.2.2) and Abel's test we find that

$$\sum \psi_r \qquad \text{is convergent,}$$

and so $$\sum_r^\infty \frac{\psi_\nu}{\nu} = o\left(\frac{1}{r}\right) \quad (r \to \infty),$$

and $$\sum_r^\infty \frac{\phi_\nu}{\nu} = o\left(\frac{1}{r}\right) \quad (r \to \infty).$$

It follows, therefore, that for each $\epsilon\,(>0)$ there exists a number $\rho > |n|$ depending on ϵ such that

$$\left|\sum_r^\infty \frac{\phi_\nu}{\nu}\right| < \frac{\epsilon}{r} \qquad \text{for } r \geq \rho.$$

Hence by Abel's partial summation method

$$\sum_\sigma^\tau \phi_r P_M(r) = \sum_\sigma^\tau \frac{\phi_r}{r}\, r P_M(r) = \sum_\sigma^\tau \Phi_r \{r P_M(r) - (r-1) P_M(r-1)\} +$$
$$+ \Phi_\sigma (\sigma-1) P_M(\sigma-1) - \Phi_{\tau+1}\, \tau P_M(\tau),$$

where $$\Phi_r = \sum_r^\infty \frac{\phi_\nu}{\nu},$$

and for $\tau > \sigma \geq \rho$

$$\left|\sum_\sigma^\tau \phi_r P_M(r)\right| \leq \epsilon \left\{ \sum_\sigma^\tau \left| P_M(r) - P_M(r-1)\right| + \sum_\sigma^\tau \left|\frac{P_M(r-1)}{r}\right| + \right.$$
$$\left. + \left|P_M(\sigma-1)\right| + \left|P_M(\tau)\right| \right\},$$

and as

$$P_M(r-1) = Q_M(r-1) + \frac{r-1}{M} \sum_0^M \left\{ \frac{1}{m+r-\frac{1}{2}} + \frac{1}{m+\frac{3}{2}-r} \right\}$$

$$\frac{1}{r} \left| P_M(r-1) \right| \leq \frac{1}{r} \left| Q_M(r-1) \right| + \frac{1}{M} \left| \sum_0^M \left\{ \frac{1}{m+r-\frac{1}{2}} + \frac{1}{m-r+\frac{3}{2}} \right\} \right|.$$

It follows from Lemmas 1, 3, and 4 that

$$\left| \sum_\sigma^\tau \phi_r P_M(r) \right| \leq \epsilon K' + \epsilon \left| P_M(\tau) \right| + \epsilon \left| P_M(\sigma-1) \right|.$$

Similarly for $\tau > \sigma \geq \rho$

$$\left| \sum_\sigma^\tau \psi_r P_M(r) \right| \leq \epsilon K'' + \epsilon \left| P_M(\tau) \right| + \epsilon \left| P_M(\sigma-1) \right|.$$

Thus

$$|J_M| \leq \left| \sum_1^{\rho}{}' \left\{ \phi_r + \psi_r \right\} P_M(r) \right| + \lim_{T\to\infty} \left| \sum_\rho^T \left\{ \phi_r + \psi_r \right\} P_M(r) \right|$$

$$\leq \left| \sum_1^{\rho}{}' \left\{ \phi_r + \psi_r \right\} P_M(r) \right| + 2\epsilon \left| P_M(\rho-1) \right| + \epsilon K.$$

Now as $M \to \infty$, $P_M(r) \to 0$ for $1 \leq r \leq \rho$, and so

$$\overline{\lim_{M\to\infty}} \left| J_M \right| \leq K\epsilon. \qquad \ldots\ldots(1.4.3)$$

Similarly let

$$L_M = \lim_{T\to\infty} \left(\sum_{-T}^{-1} + \sum_1^T \right)' \frac{a_r}{n-r} I_M(r) \qquad (r \neq n)$$

$$= \lim_{T\to\infty} \sum_1^T{}' \left\{ \phi_r' + \psi_r' \right\} I_M(r),$$

where $\phi_r' = \dfrac{n(a_r + a_{-r})}{n^2 - r^2}$, $\psi_r' = \dfrac{r(a_r - a_{-r})}{n^2 - r^2}$.

Here also by (1.2.1), (1.2.2) and Abel's test both

$$\sum \phi'_\nu \quad \text{and} \quad \sum \psi'_\nu \qquad \text{are convergent.}$$

Using again the Abel's partial summation method

$$\sum_\sigma^\tau \phi'_\nu I_M(\nu) = \sum_\sigma^\tau \Phi'_\nu \{ I_M(\nu) - I_M(\nu-1) \} + \Phi'_\sigma (\sigma-1) I_M(\sigma-1) - \Phi'_{\tau+1} \tau I_M(\tau),$$

where $$\Phi'_\nu = \sum_\nu^\infty \phi'_\nu ,$$

and so for $\tau > \sigma \geq \rho$, where $\rho = \rho(\epsilon)$ is such that $|\Phi'_\nu| < \epsilon$ for $r \geq \rho$

$$\left| \sum_\sigma^\tau \phi'_\nu I_M(\nu) \right| \leq \epsilon \left\{ \sum_\sigma^\tau |I_M(\nu) - I_M(\nu-1)| + |I_M(\sigma-1)| + |I_M(\tau)| \right\}$$

$$\leq \epsilon k' M + \epsilon |I_M(\sigma-1)| + \epsilon |I_M(\tau)| \text{ (by Lemma 5).}$$

Similarly

$$\left| \sum_\sigma^\tau \psi'_\nu I_M(\nu) \right| \leq \epsilon k'' M + \epsilon |I_M(\sigma-1)| + \epsilon |I_M(\tau)|$$

and as $\frac{1}{M} I_M(\nu) \to 0$ as $M \to \infty$ for $1 \leq r \leq \rho$, we get, as in the case of J_M, that

$$\lim_{M \to \infty} \frac{1}{M} |L_M| \leq k\epsilon . \qquad \ldots\ldots(1.4.4)$$

Combining (1.4.3) and (1.4.4) we get the required result.

(1.5) REMARK. Condition (1.2.2) will still be satisfied if we have the convergence of $\sum_1^\infty \frac{a_n + a_{-n}}{n}$ and so the above result of the

summability of the series $\sum_{-\infty}^{\infty} \frac{b_n}{n+m+\frac{1}{2}}$, in the restricted sense, to the 'sum' πa_m will still hold good. But now the series $\sum_{-\infty}^{\infty} \frac{a_n}{n+m+\frac{1}{2}}$ will converge in the ordinary sense, and we thus get an alternative proof to Titchmarsh [2], Theorem 1 (cf. -(1.1),p.8).

(1.6) THEOREM. If the two sequences $\{a_n\}$ and $\{b_n\}$ are as defined in Theorem (1.2), then for $p > 1$

$$\bar{M}\left(|b_n|^p\right) \leq N_p^p \bar{M}\left(|a_n|^p\right)$$

where $$\bar{M}(a_n) = \overline{\lim_{N\to\infty}} \frac{1}{2N} \sum_{-N}^{N} a_n ,$$

and N_p is the constant of Theorem (C).

Proof. Let $\alpha_n = a_n \quad (-c \leq n \leq c)$

$= 0 \quad$ elsewhere,

$a_n = \alpha_n + A_n$

and let β_n and B_n be the transform (1.1.1) of α_n and A_n respectively, then obviously

$$b_n = \beta_n + B_n, \qquad \ldots\ldots(1.6.1)$$

and $$\sum_{-\infty}^{\infty} |\beta_n|^p \qquad (p > 1)$$

is convergent such that by Theorem (C)

$$\sum_{-\infty}^{\infty} |\beta_n|^p \leq N_p^p \sum_{-\infty}^{\infty} |\alpha_n|^p \qquad (p>1).$$

Hence

$$\sum_{-c}^{c} |\beta_n|^p \leq \sum_{-\infty}^{\infty} |\beta_n|^p \leq N_p^p \sum_{-\infty}^{\infty} |\alpha_n|^p = N_p^p \sum_{-c}^{c} |\alpha_n|^p = N_p^p \sum_{-c}^{c} |a_n|^p . \qquad \ldots\ldots(1.6.2)$$

Now by (1.1.1)

$$B_m = \frac{1}{\pi} \sum_{c+1}^{\infty} \frac{(m+\frac{1}{2})(a_n + a_{-n}) - n(a_n - a_{-n})}{(m+\frac{1}{2})^2 - n^2},$$

and so by (1.2.1), (1.2.2) and Abel's test

$$\begin{aligned} |B_m| &\leq (m + \tfrac{1}{2})o\left(\frac{1}{c}\right) + o(1) \\ &\leq m\,o\left(\frac{1}{c}\right) + \tfrac{1}{2}o\left(\frac{1}{c}\right) + o(1) \\ &\leq c\,o\left(\frac{1}{c}\right) + \tfrac{1}{2}o\left(\frac{1}{c}\right) + o(1) \qquad m \leq c. \end{aligned}$$

Hence we may write

$$|B_m| \leq U_c, \qquad (-c \leq m \leq c)$$

where U_c tends to zero as $c \longrightarrow \infty$.

Now using Minkowski's inequality on (1.6.1) we get

$$\left(\frac{1}{2c}\sum_{-c}^{c} |b_n|^p\right)^{\frac{1}{p}} \leq \left(\frac{1}{2c}\sum_{-c}^{c} |\beta_n|^p\right)^{\frac{1}{p}} + \left(\frac{1}{2c}\sum_{-c}^{c} |B_n|^p\right)^{\frac{1}{p}}$$

$$\leq N_p \left(\frac{1}{2c}\sum_{-c}^{c} |a_n|^p\right)^{\frac{1}{p}} + U_c$$

(by (1.6.2)).

Letting $c \longrightarrow \infty$, we have

$$\left[\bar{M}\left(|b_n|^p\right)\right]^{\frac{1}{p}} \leq \left[N_p^p\, \bar{M}\left(|a_n|^p\right)\right]^{\frac{1}{p}}.$$

Hence the result.

CHAPTER II

THE GOLDBERG TRANSFORM

(2.1) The transform

$$f^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \frac{\sin t}{t^2} dt$$

as indicated in the Introduction, is called here the Goldberg transform. If f(x) belongs to L^2 then it was shown by Goldberg [5] that so does $f^*(x)$, and if F(x) and $\mathcal{F}(x)$ be the Fourier transforms(+) of f and f^* respectively, then

$$\mathcal{F}(x) = -i \operatorname{sgn} x \, F(x) \{1 - \Delta(x)\} \quad \ldots.(2.1.1)$$

where

$$\Delta(x) = \begin{cases} 1 - |x| & (|x| \le 1) \\ 0 & (|x| > 1) \end{cases} \quad \ldots.(2.1.2)$$

is the Fourier transform of $\delta(x)$ (the function defined by (0.10). (2.1.1) may also be written as

$$\mathcal{F}(x) = \begin{cases} -ix \, F(x) & (|x| \le 1) \\ -i \operatorname{sgn} x \, F(x) & (|x| > 1) \end{cases} \quad \ldots.(2.1.3)$$

and thus

$$\frac{\mathcal{F}(x)}{x} \in L^2(-1, 1). \quad \ldots.(2.1.4)$$

Heywood [6] showed that (2.1.4) is a necessary and sufficient condition for $f^*(x)$ to be the Goldberg transform of an L^2 function. In fact (2.1.3) may be used to establish a more fundamental result, namely

(2.2) THEOREM. If f(x) belongs to L^2 and $f^*(x)$ be its Goldberg

---

(+). Capital letters, representing functions, will henceforth denote Fourier transforms of functions given by corresponding small letters.

transform, then

$$f^*(x) = \phi'(x)$$

where $\phi(x)$ and $\phi'(x)$ are functions of L^2, and

$$\Phi(x) = \begin{cases} F(x) & (|x| \le 1) \\ \frac{1}{|x|} F(x) & (|x| > 1). \end{cases}$$

Proof: As $\mathcal{F}(x) \in L^2$ and $\frac{\mathcal{F}(x)}{x} \in L^2(-1, 1)$,

$\frac{\mathcal{F}(x)}{x}$ belongs to both L and L^2.

Let us write then

$$\frac{i\,\mathcal{F}(x)}{x} = \Phi(x)$$

so that $$\mathcal{F}(x) = -ix\,\Phi(x).$$

Let $\phi(x)$ be the inverse Fourier transform of $\Phi(x)$.

Then by Theorem (M), (0.14)

$$f^*(x) = \phi'(x).$$

Again by (2.1.3)

$$\Phi(x) = \begin{cases} F(x) & (|x| \le 1) \\ \frac{1}{|x|} F(x) & (|x| > 1). \end{cases}$$

This completes the proof of the theorem.

On taking the inverse Fourier transforms of both sides and remembering that the Fourier transform of

$$\sqrt{\frac{2}{\pi}}\ \frac{\sin x}{x}$$

is

$$\begin{cases} 1 & (|x| \le 1) \\ 0 & (|x| > 1), \end{cases}$$

and that of

$$\sqrt{\frac{2}{\pi}} \int_1^\infty \frac{\cos ut}{u}\, du$$

is

$$\left.\begin{array}{ll} 0 & (|x| \le 1) \\ \frac{1}{|x|} & (|x| > 1) \end{array}\right\}$$

we obtain by Theorem (L)

$$\phi(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\sin t}{t} + \int_1^\infty \frac{\cos ut}{u}\, du \right\} dt. \quad \ldots.(2.2.1)$$

Inversely we may write

$$\left.\begin{array}{rll} F(x) = & \Phi(x) & (|x| \le 1) \\ & |x|\, \Phi(x) & (|x| > 1) \end{array}\right\}$$

$$= \Phi(x)\, \Delta(x) + |x|\, \Phi(x).$$

And as $-|x|\, \Phi(x)$ is the Fourier transform of the Hilbert transform of $\phi'(x)$ (cf. (0.3) & (0.14)), we get on using Theorem(L) again

$$f = \phi .o. \delta - H\phi' \qquad \ldots.(2.2.2)$$

where $H\phi'$ is the Hilbert transform of ϕ'.

Remark. If ϕ, ϕ' and γ, γ' satisfy the conditions of Theorem (M), then it could be seen easily that

(i) $$\lim_{\substack{\xi \to \infty \\ \eta \to \infty}} \int_{-\eta}^{\xi} \phi'(x)\, dx = 0$$

(ii) $$\int_{-\infty}^{\infty} \phi(x)\, \gamma'(x)\, dx = -\int_{-\infty}^{\infty} \gamma(x)\, \phi'(x)\, dx.$$

Theorems (I) & (J), i.e. Heywood's theorems 2, and 3, follow at once from (i) and (ii) above and Theorem (2.2).

(2.3) THEOREM. If ϕ be absolutely continuous, and ϕ and ϕ' both belong to L^2, then ϕ' is the Goldberg transform of an L^2 function.

Proof: Now by (0.13)

$$\int_{-\infty}^{\infty} \phi'(t)\, \operatorname{sgn}(t-x)\, dt = -2\phi(x). \quad \ldots.(2.3.1)$$

Moreover, as ϕ' belongs to L^2

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \phi'(t)\, \frac{dt}{t-x} \quad \ldots.(2.3.2)$$

represents the Hilbert transform of ϕ', and so by Theorem (A) belongs to L^2. Again the two integrals

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \phi'(t)\, \frac{1-\cos(t-x)}{t-x}\, dt \text{ and } \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \phi'(t) \int_{1}^{\infty} \frac{\sin(t-x)u}{u}\, du\, dt$$

represent convolutions of L^2 functions, and so by Theorem (L), their Fourier transforms are

$$\left.\begin{array}{ll} -|x|\,\Phi(x) & (|x|\le 1) \\ 0 & (|x|>1) \\ 0 & (|x|\le 1) \\ -\Phi(x) & (|x|>1) \end{array}\right\} \quad \ldots.(2.3.3)$$

and

respectively (the Fourier transform of ϕ' is $-ix\Phi$ by (0.14), and those of the other two functions in the convolutions may be found in the Appendix).

Thus we see that

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \phi'(t) \left\{ \frac{1}{t-x} - \frac{1-\cos(t-x)}{t-x} + \frac{\pi}{2} \operatorname{sgn}(t-x) - \int_{1}^{\infty} \frac{\sin(t-x)u}{u}\, du \right\} dt$$

$$= \frac{1}{\pi} P \int_{-\infty}^{\infty} \phi'(t) \left\{ \frac{\cos(t-x)}{t-x} + \int_{0}^{1} \frac{\sin(t-x)u}{u}\, du \right\} dt \quad \ldots.(2.3.4)$$

exists for almost all x, and belongs to L^2.

Let us denote (2.3.4) by f(x) with F(x) as its Fourier transform. Hence by (2.3.1), (2.3.2) and (2.3.3)

$$F(x) = \begin{cases} -\Phi(x) & (|x| \le 1) \\ -|x|\,\Phi(x) & (|x| > 1), \end{cases}$$

and so by Theorem (2.2)

$$f^*(x) = -\phi'(x)$$

or
$$\phi(x) = \int_x^\infty f^*(t)\,dt. \qquad \ldots.(2.3.5)$$

This shows that the class of Goldberg transforms of functions of L^2 is the class of functions φ of L^2 such that

$$\int_x^\infty \varphi(t)\,dt$$

is also of L^2.

(2.4) THEOREM. A necessary and sufficient condition for

$$f^*(x) = f'(x),$$

f being a function belonging to L^2, is that $F(x) = 0$ (a.e) in $|x| > 1$

Proof: Necessity.

Let
$$f^*(x) = f'(x)$$

f' belongs to L^2 since f^* belongs to L^2, then by (0.14) and (2.1.1)

$$-i\,\mathrm{sgn}\,x\,F(x)\{1-\Delta(x)\} = -ix\,F(x).$$

Of course the two sides are both equal to $-ix\,F(x)$ $(|x| \le 1)$,

but for $|x|>1$, the L.H.S. is $-i\,\mathrm{sgn}x\,F(x)$, and so

$$F(x) = 0 \quad (a.e) \quad \text{in} \quad |x|>1.$$

Sufficiency.

Let $$F(x) = 0 \quad (a.e) \quad \text{in} \quad |x|>1$$

so as above $$-i\,\mathrm{sgn}x\,F(x)\{1-\Delta(x)\} = -ixF(x),$$

or $$f^*(x) = f'(x).$$

It is obvious here that $xF(x)$ belongs to L^2, and so $f'(x)$ also belongs to L^2.

Corollary 1. If $f_r(x)$, $(r = 1,2,3,\ldots\ldots)$, be a sequence of L^2 functions such that (i) $f_r'(x)$ also belongs to L^2, $(r = 2,3,\ldots\ldots\ldots)$ i.e. not necessarily for $r = 1$, and

(ii) $$f_r^*(x) = f_{r+1}'(x), \quad (r = 1,2,3,\ldots.),$$

and if $F_r(x)$ be the Fourier transform of $f_r(x)$, then for $|x| \le 1$

$$F_r(x) = F_s(x) \quad (a.e)$$

for all positive integral values of r and s.

Corollary 2. If in the above functions, for one value of r

$$f_r^* = g_r,$$

where g_r denotes the Hilbert transform of f_r, (+), then it is true for all r.

(+). g will represent the Hilbert transform of f throughout this thesis.

(2.5) Alternative proof of Theorem (H).

Here we make use of the fact that the two functions

$$\text{(i)} \qquad \frac{1-\cos t}{t}\ ,$$

$$\text{(ii)} \qquad \int_1^\infty \frac{\sin tu}{u}\,du$$

are such that (i) satisfies the conditions of Theorem (2.4) because its Fourier transform vanishes for $|x|>1$, and as such its Goldberg transform is the same as its derivative, namely

$$\frac{\sin t}{t} - \frac{1-\cos t}{t^2}\ .$$

It is of course known that the Hilbert transform of (i) is

$$\frac{\sin t}{t}\ .$$

Function (ii), on the other hand, is such that its Fourier transform vanishes on $|x|\le 1$, and so satisfies the conditions of Theorem (D). Its Goldberg and Hilbert transforms are therefore, by (0.9), equal. Hence

$$\frac{1-\cos t}{t^2} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos u}{u}\left\{\frac{u-t-\sin(u-t)}{(u-t)^2}\right\}dt,$$

and

$$0 = \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\int_1^\infty \frac{\sin uv}{u}\,dv\left\{\frac{u-t-\sin(u-t)}{(u-t)^2}\right\}\right]du,$$

and so, say, I = f.o.δ

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{1-\cos(t-x)}{(t-x)^2}\,dt = \frac{1}{\pi^2}\int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty}\left[\frac{1-\cos u}{u} + \int_1^\infty \frac{\sin uv}{v}\,dv\right]\frac{u-t+x-\sin(u-t+x)}{(u-t+x)^2}\,du.$$

Now as

$$\int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} \phi(u)\,\psi(u-t)\,du = \lim_{a\to\infty} \int_{-\infty}^{\infty} f(t)\,dt \int_{-a}^{a} \phi(u)\,\psi(u-t)\,du$$

$$= \lim_{a\to\infty} \int_{-a}^{a} \phi(u)\,du \int_{-\infty}^{\infty} f(t)\,\psi(u-t)\,dt = \int_{-\infty}^{\infty} \phi(u)\,du \int_{-\infty}^{\infty} f(t)\,\psi(u-t)\,dt$$

if f, ϕ, ψ and $\int_{-\infty}^{\infty} \phi(u)\,\psi(u-t)\,du$ all belong to L^2,

$$I = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \left\{ \frac{1-\cos u}{u} + \int_1^{\infty} \frac{\sin uv}{v}\,dv \right\} du \int_{-\infty}^{\infty} f(t)\,\frac{u-t+x-\sin(u-t+x)}{(u-t+x)^2}\,dt$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left\{ \frac{1-\cos u}{u} + \int_1^{\infty} \frac{\sin uv}{v}\,dv \right\} \left\{ f^*(u+x) - g(u+x) \right\} du.$$

Now by Theorem (L) the Fourier transform of

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1-\cos u}{u}\, g(u+x)\,du$$

is given by

$$\left.\begin{array}{ll} -\,i\,\mathrm{sgn}x\; G(x) & (|x| \le 1) \\ 0 & (|x| > 1), \end{array}\right\}$$

and that of

$$\frac{1}{\pi} \int_{-\infty}^{\infty} g(u+x) \int_1^{\infty} \frac{\sin uv}{v}\,dv$$

is given by

$$\left.\begin{array}{ll} 0 & (|x| \le 1) \\ -\,\dfrac{i\,G(x)}{x} & (|x| > 1). \end{array}\right\}$$

Hence adding the two we find that the Fourier transform of

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left\{\frac{1-\cos u}{u}+\int_{1}^{\infty}\frac{\sin uv}{v}dv\right\}g(u+x)\,du$$

is given by

$$\left.\begin{array}{ll} -\,i\,\mathrm{sgn}x\;G(x) & (|x|\le 1)\\ -\,\dfrac{i\,G(x)}{x} & (|x|>1)\end{array}\right\}$$

or by (0.3)

$$\left.\begin{array}{ll} -\,F(x) & (|x|\le 1)\\ -\,\dfrac{F(x)}{|x|} & (|x|>1).\end{array}\right\}$$

Hence by (2.3.5)

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left\{\frac{1-\cos u}{u}+\int_{1}^{\infty}\frac{\sin uv}{v}dv\right\}g(u+x)\,du =$$

$$= \int_{x}^{\infty} f^{*}(u)\,du$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} f^{*}(u)\,\mathrm{sgn}(u-x)\,du.$$

Thus $$I = \frac{1}{\pi}\int_{-\infty}^{\infty} f^{*}(u)\left\{\frac{1-\cos(u-x)}{u-x}-\frac{\pi}{2}\,\mathrm{sgn}(u-x)+\int_{1}^{\infty}\frac{\sin(u-x)v}{v}dv\right\}du$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f^{*}(u)\left\{\frac{1-\cos(u-x)}{u-x}-\int_{0}^{1}\frac{\sin(u-x)v}{v}dv\right\}du. \quad (2.5.1)$$

But by Theorem (E), by replacing f by g, since it is true for all L^2 functions

$$I = f.o.\,\delta = f + g^{*} \qquad \ldots.(2.5.2)$$

where g^{*} is the Goldberg transform of g.

If now $\mathcal{G}$ be its Fourier transform, then by (2.1.1)

$$\begin{aligned}\mathcal{G}(x) &= -i\,\mathrm{sgn}\,x\,G(x)\{1-\Delta(x)\}\\ &= -F(x)\{1-\Delta(x)\} \qquad \text{(by (0.3))}\\ &= -i\,\mathrm{sgn}\,x\,\mathcal{F}(x), \qquad \text{(again by (2.1.1))}\end{aligned}$$

and so

$$g^*(x) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{f^*(u)}{u-x}\,du. \qquad \ldots.(2.5.3)$$

Combining (2.5.1), (2.5.2) and (2.5.3) we get (0.12), and this completes the proof.

(2.6) THEOREM. If f(x), f'(x) and f"(x) belong to L^2, and $f^*(x)$ be the Goldberg transform of f(x), then

$$f^* = f' + f'.o.\bar{\sigma} - f".o.\sigma. \qquad \ldots.(2.6.1)$$

Where $\sigma(x) = \sqrt{\frac{2}{\pi}}\int_1^{\infty} \frac{\sin ux}{u}\,du$

and $\bar{\sigma}(x)$ is the Hilbert transform of $\sigma(x)$.

Proof: As in (2.1.3)

$$\mathcal{F}(x) = \left.\begin{matrix} -ix\,F(x) & (|x|\le 1)\\ -i\,\mathrm{sgn}\,x\,F(x) & (|x|>1),\end{matrix}\right\}$$

or $$\mathcal{F}(x) = -ix\,F(x) - ixF(x)\{-i\,\mathrm{sgn}\,x\,\Sigma(x)\} + \Sigma(x)x^2F(x) \ldots.(2.6.2)$$

where

$$\Sigma(x) = \left.\begin{matrix} 0 & (|x|\le 1)\\ \frac{i}{x} & (|x|>1)\end{matrix}\right\}$$

is the Fourier transform of $\sigma(x)$.

By (0.14) the Fourier transforms of f'(x) and f"(x) are

$$- i x F(x) \qquad \text{and} \qquad - x^2 F(x)$$

respectively. On taking the inverse Fourier transform of (2.6.2) and using Theorem (L) we get (2.6.1).

(2.7) THEOREM. If f and f' belong to L^2, then

$$(f^*)' = (f')^*$$

Proof: As in (2.6) the Fourier transform of f' is $- i x F(x)$, and so by (2.1.1) the Fourier transform of $(f')^*$ is

$$-|x| F(x) \{ 1 - \Delta(x) \}.$$

Again by (0.14) the Fourier transform of $(f^*)'$ is

$$-ix \left[-i \operatorname{sgn} x F(x) \{ 1 - \Delta(x) \} \right].$$

Hence the result. We also observe here that $(f^*)'$ and $(f')^*$ belong to L^2.

(2.8) THEOREM. If f satisfies the conditions of the above theorem, then

$$f' = - (g^*)' - \sum g^* .o. \delta_k^*$$

the series on the right converging in the L^2 mean, and δ_k are as defined in (0.11).

Proof: By (2.7) $(g^*)'$ belongs to L^2, since by (2.5.3) g^* is also the Hilbert transform of f^*. As in (2.5) the Fourier transform of g^* is

$$- i \operatorname{sgn} x \, \mathcal{F}(x)$$

and so by (0.14) the Fourier transform of $(g^*)'$ is $- |x| \mathcal{F}(x)$.

Hence $|x|\,\mathcal{F}(x)$ belongs to L^2.

Now by (2.1.1) $\quad -ix\,F(x) = \dfrac{|x|\,\mathcal{F}(x)}{1-\Delta(x)} = |x|\,\mathcal{F}(x)\sum_{k=0}^{\infty}\Delta^k$

the series is convergent for $x \neq 0$, since then $|\Delta| \leq 1$

Hence $-ix\,F(x) = |x|\,\mathcal{F}(x) - (-i\,\mathrm{sgn}\,x\,\mathcal{F}(x))\left[-ix\Delta - ix\Delta^2 \cdots\right]$ .....(2.8.1)

Now as Δ^k, being the Fourier transform of δ_k, vanishes for $|x| > 1$, hence by Theorem (2.4)

$$\delta_k^* = \delta_k' \qquad \text{for all k.}$$

On taking the inverse Fourier transforms of (2.8.1) and following the pattern of the proof of Theorem (F) we get the required result.

(2.9) REMARK. Theorems (2.6) and (2.8) are in a way analogous to Theorems (E) & (F). There f^* is expressed in terms of g, and then an expression is found to give g in terms of f^*. Here we have tried to find expressions containing f^* or g^* and the derived function f'. We have shown in (2.4) that, when $F(x) = 0$ (a.e) (+) for $|x| > 1$, then

$$f^* = f'$$

a result analogous to Theorem (D), namely $f^* = g$ when $F(x) = 0$ (a.e) for $|x| \leq 1$.

(+). Functions, involved in our discussions of Chapters 2 - 7, exist, in general, almost everywhere and as such relations between them also exist almost everywhere. The phrase 'almost everywhere' is, therefore in general, dropped. Nevertheless I must admit that I have not been quite consistent in this respect.

CHAPTER III

THE BOAS TRANSFORM

(3.1) The Goldberg transform

$$f^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \frac{\sin t}{t^2} dt$$

discussed in the preceding chapter is in fact a particular case, as mentioned in the Introduction, of a more general transform where $\frac{\sin t}{t}$ in the integrand is replaced by a general function H(t) (+) satisfying the following conditions

$$\text{(i)} \quad H(t) = \int_{-1}^{1} \cos st \, d\omega(s),$$

$\omega(x)$ being a continuous function of bounded variation on (-1,1) and

$$\text{(ii)} \quad \frac{1}{\pi} \int_{-\infty}^{\infty} f(x+t) \frac{H(t)}{t} dt \qquad \ldots\ldots(3.1.1)$$

exists whenever

$$\frac{1}{\pi} \int_{-\infty}^{\infty} f(x+t) \frac{dt}{t}$$

exists. This transform, which we call the Boas transform, was first discussed by Boas [4]. For his Theorem 2 (Theorem (D) in the Introduction), where f(x) belongs to L^2 and so

$$\int_{-\infty}^{\infty} f(x+t) \frac{dt}{t}$$

exists in Cauchy's sense, Boas subjected H(t) to yet another set

---

(+). Contrary to our earlier footnote about Capitals H(t) is not the Fourier transform of h(t).

of conditions, namely

$$H(0) = 1$$

and $\omega(\delta) - \omega(-\delta) = \pm 1$ almost nowhere.

We try to establish here results similar to Theorems (E), (G), (H), (I) and (J) in the general case with H(t) instead of $\frac{\sin t}{t}$, where

$$H(t) = \cos t + t \int_0^1 \sin tu \, h(u) \, du, \qquad \ldots\ldots(3.1.2)$$

h(u) being an odd continuous function of bounded variation on (0,1) such that $h(1) = 1$. $\ldots\ldots(3.1.3)$

That this definition of H(t) is equivalent to that given by Boas [4] for Theorem (D) can be seen as follows:-

By the properties of Riemann - Stieltjes integrals

$$\int_{-1}^{1} \cos ut \, d\omega(u) - t \int_{-1}^{1} \sin tu \, \omega(u) \, du = \cos t,$$

since according to Boas $H(0) = 1$, and so $\omega(1) - \omega(-1) = 1$

Hence

$$H(t) = \cos t + t \int_0^1 \sin tu \, \{\omega(u) - \omega(-u)\} \, du.$$

Replacing now, $\omega(u) - \omega(-u)$ by h(u) with the condition that h(u) is to be an odd function we get (3.1.2).

If now, following Boas, we denote (3.1.1) by $f^*(x)$, then

$$f^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 h(u) \sin ut \, du \right\} dt \qquad \ldots\ldots(3.1.4)$$

It could be seen now that (3.1.4) gives the Goldberg transform if

$$h(u) = u,$$

and when $h(u) = 1$

$$\int_0^t \sin tu\, du = \frac{1-\cos t}{t},$$

(3.1.4) reduces to

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t)\, \frac{dt}{t},$$

which represents $g(x)$. Now as $g(x)$ also belongs to L^2 and as such satisfies the main feature of the General Boas transform, we are justified in calling the Hilbert transform also a particular case of (3.1.4) although here $h(u)$ does not strictly satisfy the conditions (3.1.3). Similarly if

$$h(u) = 0$$

we find that the conditions (3.1.3) are again not satisfied yet (3.1.4) exists almost everywhere and belongs to L^2. That it is so we show in (3.2).

(3.2) THEOREM. If $f(x)$ belongs to L^2, then

$$f^{\circ}(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t)\, \frac{\cos t}{t}\, dt \qquad \ldots.(3.2.1)$$

exists for almost all x, and belongs to L^2, and a necessary and

sufficient condition for

$$F(x) = 0 \quad (a.e) \quad \text{on } |x| \leq 1$$

is that $f^{\circ}(x) = g(x)$.

Similarly a necessary and sufficient condition for

$$F(x) = 0 \quad (a.e) \quad \text{for } |x| > 1$$

is that $f^{\circ}(x) = 0$.

Proof. The existence of (3.2.1) for almost all x can be proved by considering

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \frac{dt}{t}$$

and

$$\frac{1}{\pi} \int_{-\infty}^{\infty} f(x+t) \frac{1-\cos t}{t} dt,$$

the former represents g(x) for almost all x, and the latter is an absolutely convergent integral, and so

$$f^{\circ}(x) = g(x) - \frac{1}{\pi} \int_{-\infty}^{\infty} f(x+t) \frac{1-\cos t}{t} dt.$$

If now $\mathcal{F}^{\circ}(x)$ be the Fourier transform of $f^{\circ}(x)$ (it has a Fourier transform since the R.H.S. has one), then by Theorem (L)

$$\mathcal{F}^{\circ}(x) = \begin{cases} G(x) + i \operatorname{sgn} x \, F(x) & (|x| \leq 1) \\ G(x) & (|x| > 1) \end{cases}$$

$$= \begin{cases} 0 & (|x| \le 1) \\ -i\,\mathrm{sgn}\,x\,F(x) & (|x| > 1) \end{cases} \quad \text{(by (0.3))....(3.2.2)}$$

Hence $\mathcal{F}^\circ$ and consequently f° belong to L^2.

If now $F(x) = 0$ (a.e) on $|x| \le 1$,

$$\mathcal{F}^\circ(x) = G(x)$$

and so
$$f^\circ(x) = g(x). \qquad \text{....(3.2.3)}$$

Conversely if (3.2.3) holds then

$$\mathcal{F}^\circ(x) = G(x)$$

and since for $|x| \le 1$

$$\mathcal{F}^\circ(x) = G(x) + i\,\mathrm{sgn}\,x\,F(x),$$

the second term on the right must be zero almost everywhere on $(-1,1)$

If on the other hand

$$F(x) = 0 \text{ (a.e) for } |x| > 1,$$

$$\mathcal{F}^\circ(x) = 0$$

and so
$$f^\circ(x) = 0. \qquad \text{....(3.2.4)}$$

The converse can now be proved by retracing the steps backwards. (0.9) becomes now an immediate corollary to the above theorem, for now (3.1.4) can be put as

$$f^*(x) = f^\circ(x) + \frac{1}{\pi}\int_{-\infty}^{\infty} f(x+t)\int_0^{\prime} k(u)\sin ut\,du\,dt. \qquad \text{....(3.2.5)}$$

Now as

$$\sqrt{\frac{2}{\pi}}\int_0^1 h(u)\sin ut\,du \quad \varepsilon \quad L^2,$$

since its Fourier transform (+) is

$$\left.\begin{array}{ll} i\,\mathrm{sgn}\,x\; h(|x|) & (|x|\le 1) \\ 0 & (|x|>1) \end{array}\right\}$$

the second term on the right of (3.2.5), being the convolution of

$$f(x) \qquad \text{and} \qquad -\sqrt{\frac{2}{\pi}}\int_0^1 h(u)\sin ut\,du$$

vanishes for almost all x if F(x) = 0 (a.e) on $|x| \le 1$, and so by (3.2.3)

$$f^*(x) = f^\circ(x) = g(x).$$

If on the other hand F(x) = 0 (a.e) for $|x| > 1$, then by (3.2.4)

$$f^*(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x+t)\int_0^1 h(u)\sin ut\,du\,dt \cdots\cdots (3.2.6)$$

If $\mathcal{F}(x)$ be the Fourier transform of $f^*(x)$, then

$$\mathcal{F}(x) = -i\,\mathrm{sgn}\,x\; h(|x|)\, F(x).$$

In case of the Goldberg transform the above equation reduces to

$$\mathcal{F}(x) = -ix\, F(x),$$

(+). The actual calculation is on p. 46.

since here $h(x) = x$, and so by (0.14)

$$f^*(x) = f'(x),$$

which, therefore, gives another proof of Theorem (2.4).

(3.3) The extensions of the theorems of Goldberg and Heywood to cover the general case of the Boas transform (3.1.4) would be discussed in (3.4) and the following sections. Here, however, we give a third proof of Theorem (H), for on this depend the discussions of (3.4).

The third proof of Theorem (H).

By (2.1.3)

$$F(x) = \left.\begin{array}{ll} \dfrac{i\,\mathcal{F}(x)}{x} & (|x| \le 1) \\ i \operatorname{sgn} x\, \mathcal{F}(x) & (|x| > 1) \end{array}\right\} \quad \ldots\ldots (3.3.1)$$

Let $K(x)$ be the Fourier transform of

$$k(x) = \sqrt{\frac{2}{\pi}}\,\frac{\sin x}{x},$$

then

$$\left.\begin{array}{lll} K(x) = & 1 & (|x| \le 1) \\ = & 0 & (|x| > 1). \end{array}\right\}$$

Hence (3.3.1) can be written as

$$F(x) = i\,\frac{\mathcal{F}(x)}{x}\,K(x) + i \operatorname{sgn} x\, \mathcal{F}(x)\{1 - K(x)\}. \ldots\ldots (3.3.2)$$

Let $\phi(x)$ be the inverse Fourier transform of $\dfrac{i\,\mathcal{F}(x)}{x}$, for by (2.2) it belongs to L^2. By Theorem (L) the inverse Fourier transform of

$$\frac{i\,\mathcal{F}(x)}{x}\,K(x)$$

is therefore

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\phi(x+t)\,\frac{\sin t}{t}\,dt,$$

which on integrating by parts becomes

$$-\frac{1}{\pi}\int_{-\infty}^{\infty}\phi'(x+t)\int_{0}^{t}\frac{\sin u}{u}\,du\,dt,$$

since ϕ' belongs to L^2 and so by (0.13) $\phi(t)$ tends to zero as t tends to $\pm\infty$, and

$$\int_{0}^{t}\frac{\sin u}{u}\,du$$

is a bounded function of t. Now by Theorem (2.2)

$$\phi'(x) = f^{*}(x),$$

and so the above expression reduces to

$$-\frac{1}{\pi}\int_{-\infty}^{\infty}f^{*}(x+t)\int_{0}^{t}\frac{\sin ut}{u}\,du\,dt.$$

Again by (3.2.2) the inverse Fourier transform of

$$i\,\mathrm{sgn}\,x\;\mathcal{F}(x)\{1-K(x)\}$$

is given by

$$-\frac{1}{\pi}P\int_{-\infty}^{\infty}f^{*}(x+t)\,\frac{\cos t}{t}\,dt.$$

Hence combining the two we get

$$-f(x) = \frac{1}{\pi} P\int_{-\infty}^{\infty} f^*(x+t)\left\{\frac{\cos t}{t} + \int_0^{\prime} \frac{\sin tu}{u}\,du\right\}dt.$$

(3.4) THEOREM. If $f(x) \in L^2$, and $f^*(x)$ be as defined by (3.1.4) then $f^*(x)$ also belongs to L^2, and inversely

$$-f(x) = \frac{1}{\pi} P\int_{-\infty}^{\infty} f^*(x+t)\left\{\frac{\cos t}{t} + \int_0^{\prime} \frac{\sin tu}{k(u)}\,du\right\}dt \qquad \ldots.(3.4.1)$$

provided h(x) satisfies besides (3.1.3) the following condition

$$h(x) = x\,p(x), \qquad \ldots.(3.4.2)$$

where p(x) is continuous and of bounded variation and nowhere zero in (0,1), ~~and is such that K(x)/p(x) is zero almost everywhere in $|x|>1$~~, [K(x) ~~being~~ is the Fourier transform of

$$k(x) = \sqrt{\frac{2}{\pi}}\,\frac{\sin x}{x}\,.]$$

This restriction is placed on h(x) to guarantee the convergence of

$$\int_0^{\prime} \frac{\sin tu}{k(u)}\,du \qquad (-\infty < t < \infty),$$

which is an essential part of the inverse formula (3.4.1).

N.B. $f^*(x)$ denotes the Goldberg transform in (3.3) only; for the rest of this chapter $f^*(x)$ will stand for the general Boas transform as defined in (3.1.4).

Proof. We first show that $f^*(x)$ exists for almost all x and belongs to L^2.

Let

$$\Psi(x) = \left.\begin{array}{ll} i\,\mathrm{sgn}\,x\,h(|x|) & (|x|\leq 1) \\ 0 & (|x|>1) \end{array}\right\} \quad \ldots\ldots (3.4.3)$$

then $\Psi(x)$ belongs to L^2, and if $\psi(x)$ be its inverse Fourier transform then

$$\psi(x) = \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} \mathrm{sgn}\,t\,h(|t|)\,e^{-ixt}\,dt$$

$$= \frac{i}{\sqrt{2\pi}} \left[\int_0^1 h(t)\,e^{-ixt} - \int_0^1 h(t)\,e^{ixt}\,dt \right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 h(t) \sin xt\,dt.$$

Thus

$$-\frac{1}{\pi} \int_{-\infty}^{\infty} f(x+t) \int_0^1 h(u) \sin ut\,du\,dt$$

is the convolution of f(x) with $\psi(x)$, and so its Fourier transform, by Theorem (L), is given by

$$F(x)\,\Psi(x),$$

which belongs to L^2, since F(x) does so and $\Psi(x)$ is bounded.

Thus $f.o.\psi$ also belongs to L^2. On the other hand by (3.2)

$$f^{\circ}(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \frac{\cos t}{t} dt$$

exists for almost all x, and belongs to L^2. Again by (3.1.4)

$$f^{*}(x) = f^{\circ}(x) - f.o.\psi \qquad \ldots\ldots(3.4.4)$$

and so $f^{*}(x)$ exists for almost all x, and belongs to L^2.

If now, as usual, $\mathcal{F}(x)$ be the Fourier transform of $f^{*}(x)$, and

$$\Delta_k = \begin{cases} 1 - k(|x|) & (|x| \leq 1) \\ 0 & (|x| > 1) \end{cases} \qquad \ldots\ldots(3.4.5)$$

then by (3.2.2) and (3.4.4)

$$\mathcal{F}(x) = -i \operatorname{sgn} x \, F(x) \{ 1 - \Delta_k \} \qquad \ldots\ldots(3.4.6)$$

or

$$F(x) = \frac{i \operatorname{sgn} x \, \mathcal{F}(x)}{1 - \Delta_k},$$

and thus

$$\frac{\mathcal{F}(x)}{k(|x|)} \in L^2(-1, 1)$$

or

$$\frac{\mathcal{F}(x)}{x} \in L^2(-1, 1),$$

since by (3.4.2) h(x) = x p(x), and p(x) does not vanish anywhere in (0,1) and so in (-1,1), for p(x) is even, and thus

$$p(|x|)\,\frac{\mathcal{F}(x)}{h(|x|)} = \frac{\mathcal{F}(x)}{x} \in L^2(-1,1).$$

Following the pattern of the proof of (3.3) we get

$$F(x) = i\,\frac{\mathcal{F}(x)}{x}\cdot\frac{K(x)}{p(x)} + i\,\mathrm{sgn}\,x\,\mathcal{F}(x)\{1-K(x)\}. \quad \ldots.(3.4.7)$$

The inverse Fourier transform of the second term on the right is, as in (3.3)

$$-\frac{1}{\pi}P\int_{-\infty}^{\infty} f^*(x+t)\,\frac{\cos t}{t}\,dt.$$

Again as in the preceding section, let $\phi(x)$ be the inverse Fourier transform of

$$\frac{i\,\mathcal{F}(x)}{x},$$

and so, as there, $f^*(x) = \phi'(x)$. ....(3.4.8)

The inverse Fourier transform of K(x)/p(x) is obviously

$$\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\frac{e^{-ixt}}{p(t)}\,dt = \sqrt{\frac{2}{\pi}}\int_{0}^{1}\frac{\cos tx}{p(t)}\,dt \quad \text{(since p(x) is even),}$$

and $K(x)/p(x)$ is to be interpreted as equal to 0, a.e in $|x|>1$.

Hence by Theorem (L) the inverse Fourier transform of the first

term on the R.H.S. of (3.4.7) is given by

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\phi(x+t)\int_0^1\frac{\cos tv}{p(v)}\,dv\,dt,$$

which on integrating by parts becomes

$$-\frac{1}{\pi}\int_{-\infty}^{\infty}\phi'(x+t)\left(\int_0^t du\int_0^1\frac{\cos uv}{p(v)}\,dv\right)dt,$$

(since here $\phi(x)\longrightarrow 0$ as $x\longrightarrow\pm\infty$ by (0.13), and

$$\int_0^t du\int_0^1\frac{\cos uv}{p(v)}\,dv$$

is a bounded function of t,)

$$=-\frac{1}{\pi}\int_{-\infty}^{\infty}\phi'(x+t)\,dt\int_0^1\frac{dv}{p(v)}\int_0^t\cos uv\,du,$$

(since cos uv/p(v) is continuous in $0\le v\le 1$, $0\le u\le t$,)

$$=-\frac{1}{\pi}\int_{-\infty}^{\infty}\phi'(x+t)\,dt\int_0^1\frac{\sin tv}{k(v)}\,dv$$

$$=-\frac{1}{\pi}\int_{-\infty}^{\infty}f^*(x+t)\,dt\int_0^1\frac{\sin tv}{k(v)}\,dv \quad \text{(by (3.4.8)).}$$

Hence taking the inverse Fourier transforms of both sides of (3.4.7) we get (3.4.1).

(3.5) The conditions (3.1.3) and (3.4.2) are by no means necessary for the existence of $f^*(x)$ and for the inverse formula (3.4.1) as can be seen from the following theorem.

THEOREM. If $f(x)$ belongs to L^2, and $h(x)$ be such that in $(-1,1)$

$$h(x) \in L$$

$$p(x) = h(x)/x \text{ is bounded}$$

$$1/p(x) \in L^2,$$

and

$$\int_0^t \frac{\sin tv}{h(v)}\, dv \qquad \ldots.(3.5.1)$$

is a bounded function of t, then (3.1.4) exists for almost all x, and inversely (3.4.1) holds.

Proof. As $h(x) \in L(-1,1)$

$$\int_0^t h(u) \sin ut\, du$$

exists, and as in (3.4.3) has

$$\left.\begin{array}{ll} \sqrt{\frac{\pi}{2}}\, i \operatorname{sgn} x\, h(|x|) & (|x| \le 1) \\ 0 & (|x| > 1) \end{array}\right\}$$

as its Fourier transform, and so $f^*(x)$ exists (a.e) and belongs to L^2 (the argument being the same as in (3.4)).

To prove the inverse formula (3.4.1), let us write, as in (3.4.7)

$$F(x) = i \frac{\mathcal{F}(x)}{x} \cdot \frac{K(x)}{p(x)} + i \operatorname{sgn} x\, \mathcal{F}(x) \{1 - K(x)\}.$$

Here also

$$\frac{\mathcal{F}(x)}{x} \in L^2(-1, 1),$$

since p(x) is bounded in (-1,1).

Again as $1/p(x) \in L^2(-1,1)$ and $K(x) = 0$ (a.e) for $|x| > 1$

$$K(x)/p(x) \in L^2,$$

and so the first term on the right of (3.4.7) is the Fourier transform of the convolution of the inverse Fourier transforms of

$$\frac{i\,\mathcal{F}(x)}{x} \qquad \text{and} \qquad \frac{K(x)}{p(x)}\,.$$

Hence following the argument of (3.4) and using (3.5.1) we get the required result.

(3.6) THEOREM. (Extension of Theorem (I)). If f and f^* be as in (3.4) then

$$\int_{\to -\infty}^{\to \infty} f^*(x)\,dx = 0\,.$$

Proof. As in (3.4.8) $f^*(x) = \phi'(x)$, and as both $\phi(x)$ and $\phi'(x)$ belong to L^2.

$$\int_{\to -\infty}^{\to \infty} \phi'(x)\,dx = 0.$$

Hence the required result.

(3.7) THEOREM. (Extension of Theorem (J)). If $\psi(x)$ also belongs to L^2, and $\psi^*(x)$ be its (3.1.4) transform then

$$\int_{-\infty}^{\infty} f(x)\,\psi^*(x)\,dx = -\int_{-\infty}^{\infty} f^*(x)\,\psi(x)\,dx\,.$$

EX LIBR. UNIV. EDINBURGEN

The proof is exactly the same as given by Heywood [6] except that now we have Δ_h (as defined in (3.4.5)) instead of $\Delta(x)$.

His other two forms of the above theorem, namely

$$\int_{-\infty}^{\infty} f(x)\, f^*(x)\, dx = 0$$

and

$$\int_{-\infty}^{\infty} \{ f^*(x) \}^2 dx = - \int_{-\infty}^{\infty} f(x)\, f^{**}(x)\, dx$$

are also true here.

(3.8) THEOREM. (Extension of Theorem (E)). If f and f^* be as above then

$$f^* = g - g \circ \varphi$$

where

$$\varphi(x) = \sqrt{\frac{2}{\pi}} \frac{1}{x} \int_0^1 \sin xt \, d h(t). \qquad \ldots.(3.8.1)$$

Proof. Let Δ_h be as in (3.4.5), its Fourier transform (inverse) is therefore

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \{ 1 - h(|x|) \} e^{-ixt} dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \{ 1 - h(x) \} \cos tx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{\sin t}{t} - \int_0^1 \cos tx \, h(x) \, dx \right\}$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{t} \int_0^1 \sin xt \, dh(x) \qquad \text{(since } h(1) = 1\text{)}$$

$$= \rho(t).$$

Again by (3.4.6)

$$\mathcal{F}(x) = -i\,\mathrm{sgn}\,x\, F(x) - \{-i\,\mathrm{sgn}\,x\, F(x)\}\Delta_h$$

and so by Theorem (L)

$$f^* = g - g \circ \rho .$$

(3.9) THEOREM. (Extension of Theorem (G)). If f, f^* and ρ be as above then

$$f^{**} = -f + 2\rho \circ f - \rho \circ \rho \circ f.$$

The proof is again the same as given by Goldberg [5] except that we have now Δ_h and not $\Delta(x)$. We, however, give here an extension of the above result in the form

$$f^{(2n)*} = (-1)^n \left[f - 2n\rho \circ f + \frac{2n(2n-1)}{2!} \rho_2 \circ f - \cdots + \rho_{2n} \circ f \right] \quad \ldots (3.9.1)$$

$$f^{(2n-1)*} = (-1)^{n+1} \left[g - (2n-1) g \circ \rho + \cdots - \rho_{2n-1} \circ g \right] \quad \ldots (3.9.2)$$

where $f^{(n)*}$ is that transform of f which is obtained by n

repeated applications of the transform (3.1.4), and like (0.11)

$$\rho_1 = \rho$$
$$\rho_n = \rho_{n-1} \circ \rho \qquad (n = 2,3,\ldots\ldots).$$

Proof. Let the Fourier transform of $f^{(n)*}$ be $\mathcal{F}_{(n)}$ with $\mathcal{F}_{(1)} = \mathcal{F}$ then

$$\mathcal{F}_{(2n)}(x) = (-1)^n F(x)\{1 - \Delta_h\}^{2n} \qquad \ldots\ldots(3.9.3)$$

and

$$\mathcal{F}_{(2n-1)}(x) = (-1)^n i\,\mathrm{sgn}\,x\,F(x)\{1-\Delta_h\}^{2n-1}. \qquad \ldots\ldots(3.9.4)$$

We also know that Δ_h^n, for any positive integral value of n, is the Fourier transform of ρ_n. Hence we obtain the two results (3.9.1) and (3.9.2) by taking the inverse Fourier transforms of (3.9.3) and (3.9.4).

It could be seen now that Theorem (E) is a particular case of (3.9.2) when

$$h(u) = u$$

and

$$n = 1,$$

for in this case

$$\rho(t) = \sqrt{\frac{2}{\pi}}\frac{1}{t}\int_0^1 \sin xt\,dx = \sqrt{\frac{2}{\pi}}\frac{1-\cos t}{t^2} = \delta(t) \text{ (by (0.10))}.$$

Remark. Theorems (3.7), (3.8) and (3.9) are true even when h(u) satisfies (3.1.3) and not necessarily (3.4.2).

CHAPTER IV

THE COMPOUND BOAS TRANSFORMS

(4.1) DEFINITIONS. We define the function h(u) of (3.1.2) as the 'generating function' (g.f) of the Boas transform (3.1.4). Thus the g.fs. of the Hilbert and Goldberg transforms are respectively

$$1 \text{ and } u.$$

If $B_1, B_2, \ldots, B_n$ be some Boas transforms with $h_1(u), h_2(u), \ldots, h_n(u)$ as g.fs. then naturally

$$B_1 B_2 f, \quad B_1 B_2 B_3 f, \quad \ldots\ldots, \quad B_1 B_2 \ldots\ldots B_n f$$

will exist for almost all x and will belong to L^2. We define these transforms as Compound Boas transforms of order 2, 3, ... , and n respectively. Thus f^{**}, in the notation of Goldberg [5], is the Compound Goldberg transform of order 2, and $f^{(n)*}$ of order n.

We also define the product of the g.fs. of the individual transforms involved in a Compound Boas transform as the 'kernel' of the said Compound transform. Thus the kernel of f^{**} (again in the notation of Goldberg) is u^2, and that of

$$B_1 B_2 \ldots\ldots B_n f \qquad \ldots\ldots (4.1.1)$$

is $h_1(u)\, h_2(u) \ldots\ldots h_n(u)$.

We now show that the Compound transform (4.1.1) is commutable in the sense that the order of the Bs is immaterial. We prove this by taking the case of n = 2, for the general case would then be an

easy extension.

Let $B_1 B_2 f$ be a Compound Boas transform with $k_1(u)$ and $k_2(u)$ as their respective g.fs, and let F_2 be the Fourier transform of $B_2 f$ then by (3.4.6)

$$F_2(x) = -i \operatorname{sgn} x \, F(x) \{ 1 - \Delta_{k_2} \},$$

where

$$\Delta_{k_r} = \begin{cases} 1 - k_r(|x|) & (|x| \le 1) \\ 0 & (|x| > 1). \end{cases}$$

Now if the Fourier transform of $B_1 B_2 f$ be $\mathcal{F}$ then

$$\mathcal{F}(x) = -i \operatorname{sgn} x \, F_2(x) \{ 1 - \Delta_{k_1} \}$$

$$= -F(x) \{ 1 - \Delta_{k_1} \} \{ 1 - \Delta_{k_2} \} \quad \ldots\ldots(4.1.2)$$

It is easy to verify that we would get the same result if we begin with $B_1 f$ and finally get $B_2 B_1 f$.

We now prove a few theorems on the Compound Boas transforms, their kernels, orders and generating functions.

(4.2) THEOREM. If $f(x) \in L^2$ and $f^*(x)$ be as defined by (3.1.4), and $h(x)$ satisfies conditions (3.1.3) and (3.4.2), and if

$$\underline{\underline{f}}(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + 2 \int_0^1 \frac{k(u)}{1+k^2(u)} \sin ut \, du \right\} dt \quad \ldots\ldots(4.2.1)$$

then the g.f. of

$$- \underline{\underline{f}}^{**}(x) \quad \ldots\ldots(4.2.2)$$

is $$\frac{2\,k^3(u)}{1+k^2(u)}$$

i.e. the kernel of (4.2.2) is the same as its g.f. or

$$-\underset{=}{f}^{**}(x) = \frac{1}{\pi} P\int_{-\infty}^{\infty} f(x+t)\left\{\frac{\cos t}{t} + 2\int_0^1 \frac{k^3(u)}{1+k^2(u)}\sin ut\,du\right\}dt. \qquad \ldots.(4.2.3)$$

Proof.

$$\frac{2k(u)}{1+k^2(u)}$$

satisfies both conditions (3.1.3) & (3.4.2) and as such $\underset{=}{f}(x)$ exists almost everywhere and belongs to L^2, and moreover by (3.4)

$$-f(x) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \underset{=}{f}(x+t)\left\{\frac{\cos t}{t} + \frac{1}{2}\int_0^1 \frac{1+k^2(u)}{k(u)}\sin ut\,du\right\}dt. \qquad \ldots.(4.2.4)$$

As $f^*(x)$ also belongs to L^2, we may have

$$\underset{=}{f}^{*}(x) = \frac{1}{\pi} P\int_{-\infty}^{\infty} f^{*}(x+t)\left\{\frac{\cos t}{t} + 2\int_0^1 \frac{k(u)}{1+k^2(u)}\sin ut\,du\right\}dt.$$

Now it can be seen by taking the Fourier transforms of

$$\left(\underset{=}{f}\right)^* \qquad \text{and} \qquad \underline{\underline{(f^*)}}$$

that they are equal. Hence by (4.2.4)

$$-f^{*}(x) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \underset{=}{f}^{*}(x+t)\left\{\frac{\cos t}{t} + \frac{1}{2}\int_0^1 \frac{1+k^2(u)}{k(u)}\sin ut\,du\right\}dt$$

$$= \frac{1}{2\pi} P \int_{-\infty}^{\infty} \underline{\underline{f}}^*(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 k(u) \sin ut\, du \right\} dt +$$

$$+ \frac{1}{2\pi} P \int_{-\infty}^{\infty} \underline{\underline{f}}^*(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 \frac{\sin ut}{k(u)}\, du \right\} dt$$

$$= \frac{1}{2} \left\{ \underline{\underline{f}}^{**}(x) - \underline{\underline{f}}(x) \right\}$$

by (3.1.4), (3.4.1) and the fact that $\underline{\underline{(f^*)}} = \underline{\underline{(f)}}^*$.

Hence

$$\underline{\underline{f}}^{**}(x) = \underline{\underline{f}}(x) - 2 f^*(x)$$

$$= \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + 2 \int_0^1 \frac{k(u)}{1+k^2(u)} \sin ut\, du \right\} dt -$$

$$- \frac{2}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 k(u) \sin ut\, du \right\} dt$$

(by (4.2.1)),

$$= -\frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + 2 \int_0^1 \frac{k^3(u)}{1+k^2(u)} \sin ut\, du \right\} dt .$$

(4.3) THEOREM. All Compound Boas transforms of the same order and kernel are equivalent almost everywhere, i.e. if $B_1, B_2, \ldots B_n$ be one set of Boas transforms with $k_1, k_2, \ldots k_n$ as their g.fs. and if $B_1', B_2', \ldots B_n'$ be another set of Boas transforms with $k_1, k_2, \ldots k_n$ as their g.fs. and if

$$k_1 k_2 \ldots k_n = k_1 k_2 \ldots k_n$$

then

$$B_1 B_2 \ldots\ldots B_n f = B_1' B_2' \ldots\ldots B_n' f.$$

Proof. We take first the case when n = 2, the general case is a simple extension.

Let, as before, B_1, B_2 with h_1, h_2 as g.fs. be two Boas transforms. Let B_1' and B_2' be two other Boas transforms with k_1 and k_2 as their g.fs., such that

$$h_1(u)\, h_2(u) = k_1(u)\, k_2(u).$$

Hence

$$h_1(|u|)\, h_2(|u|) = k_1(|u|)\, k_2(|u|) \quad \ldots (4.3.1)$$

Let $\mathcal{F}$ be the Fourier transform of $B_1 B_2 f$ then by (4.1.2)

$$\left.\begin{aligned} \mathcal{F}(x) &= -F(x)\, h_1(|x|)\, h_2(|x|) && (|x| \le 1) \\ &\quad -F(x) && (|x| > 1), \end{aligned}\right\}$$

Now if $\mathcal{F}'$ be the Fourier transform of $B_1' B_2' f$ then as above

$$\left.\begin{aligned} \mathcal{F}'(x) &= -F(x)\, k_1(|x|)\, k_2(|x|) && (|x| \le 1) \\ &\quad -F(x) && (|x| > 1), \end{aligned}\right\}$$

and so by (4.3.1)

$$\mathcal{F}'(x) = \mathcal{F}(x).$$

Hence the required result.

In the general case if $\mathcal{F}$ and $\mathcal{F}'$ be the Fourier transforms

of $B_1 B_2 \ldots . B_n f$ and $B_1' B_2' \ldots . B_n' f$ respectively then

$$\left.\begin{aligned} \mathcal{F}(x) \;=\; & (-i\,\mathrm{sgn}\,x)^n F(x) \prod_{r=1}^{n} k_r(|x|) \quad (|x|\le 1) \\ & (-i\,\mathrm{sgn}\,x)^n F(x) \quad (|x|>1), \end{aligned}\right\}$$

and

$$\left.\begin{aligned} \mathcal{F}'(x) \;=\; & (-i\,\mathrm{sgn}\,x)^n F(x) \prod_{r=1}^{n} k_r(|x|) \quad (|x|\le 1) \\ & (-i\,\mathrm{sgn}\,x)^n F(x) \quad (|x|>1), \end{aligned}\right\}$$

and obviously the two are equal.

(4.4) Till now we have confined our choice of $k_\gamma(x)$ to the class of functions which satisfy conditions (3.1.3). We may, however, select under suitable conditions one or more of these $k_\gamma(x)$ in such a manner that if they do not satisfy (3.1.3) their reciprocals satisfy both set of conditions (3.1.3) and (3.4.2). But now the transform

$$B_1' B_2' \ldots\ldots B_n' f$$

will cease to be strictly commutable in so far as the positions of those B_γ' s are concerned for which $k_\gamma(x)$ are of the type discussed above. We, in fact, prove a theorem by taking only one such B_γ', which of course can be generalised without much difficulty.

THEOREM. If $f(x) \in L^2$ and $B_1, B_2, \ldots, B_n, B_1', \ldots, B_n'$ have the same meaning as in (4.3) except that B_ν' $(\nu < n)$ is such that $k_\nu(x)$

does not satisfy conditions (3.1.3) but on the other hand $\frac{1}{k_\nu(x)}$ satisfies both (3.1.3) and (3.4.2) then

$$B_1 B_2 \ldots\ldots B_n f = B_1' B_2' \ldots\ldots B_n' f$$

provided $$l_1 l_2 \ldots l_n = k_1 k_2 \ldots k_n ,$$

and

$$\frac{\mathcal{F}_\nu(x)}{x} \in L^2(-1, 1)$$

where $\mathcal{F}_\nu(x)$ is the Fourier transform of

$$B_{\nu+1}' B_{\nu+2}' \ldots\ldots B_n' f .$$

Proof. We take, as usual, the case when n = 2, and prove that

$$B_1 B_2 f = B_1' B_2' f$$

where k_1, l_2 and k_2 are the g.fs. of B_1, B_2 and B_2', and B_1' is such that $k_1(x)$ does not satisfy (3.1.3) but $\frac{1}{k_1(x)}$ satisfies both (3.1.3) and (3.4.2).

We first prove a lemma.

LEMMA. If $\phi(x) \in L^2$ and

$$\frac{\Phi(x)}{x} \in L^2(-1, 1),$$

then

$$\psi(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \phi(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 \frac{\sin ut}{h(u)} du \right\} dt$$

belongs to L^2, where h(u) satisfies both (3.1.3) and (3.4.2), and in fact

$$\phi = B\psi$$

where B is the Boas transform generated by h(u). The Fourier transform of $\psi(x)$ is given by

$$- \frac{i \operatorname{sgn} x \ \Phi(x)}{1 - \Delta_h}$$

where Δ_h is as in (3.4.5).

Proof of the lemma. By (3.4.2)

$$h(x) = x\, p(x)$$

where p(x) is of bounded variation and does not vanish anywhere in (-1,1), and so

$$\frac{\Phi(x)}{h(x)} \in L^2(-1, 1).$$

If K(x) be as in (3.3) then

$$\frac{i\, \Phi(x)}{h(x)} \cdot K(x) \in L^2.$$

Hence J(x) =

$$i \frac{\Phi(x)}{h(x)} K(x) + i \operatorname{sgn} x \, \Phi(x) \left[1 - K(x) \right] \quad \ldots\ldots (4.4.1)$$

belongs to L^2. Now as in (3.4) the inverse Fourier transform of the R.H.S. of (4.4.1)

$$= - \frac{1}{\pi} P \int_{-\infty}^{\infty} \phi(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 \frac{\sin tu}{h(u)} du \right\} dt,$$

which therefore belongs to L^2. Moreover by (4.4.1)

$$J(x) = \frac{i \operatorname{sgn} x \ \Phi(x)}{1 - \Delta_h}$$

Hence the Fourier transform of $\psi(x)$ is

$$-\frac{i\,\mathrm{sgn}\,x\,\Phi(x)}{1-\Delta_h}, \qquad \ldots.(4.4.2)$$

since $j(x) = -\psi(x)$.

Proof of the theorem. Let the Fourier transform of $B_2'f$ be $\mathcal{F}_1$ (in the general case the Fourier transform of $B'_{\nu+1}\ldots.B'_n f$ is $\mathcal{F}_\nu$) then as usual

$$\mathcal{F}_1(x) = -i\,\mathrm{sgn}\,x\,F(x)\{1-\Delta_{k_2}\},$$

where $$\Delta_{k_2} = \begin{cases} 1-k_2(|x|) & (|x|\le 1) \\ 0 & (|x|>1). \end{cases}$$

Now if $\frac{\mathcal{F}_1(x)}{x} \in L^2(-1,1)$, by the lemma proved above

$$B_1'B_2'f = \frac{1}{\pi}P\int_{-\infty}^{\infty} B_2'f(x+t)\left\{\frac{\cos t}{t} + \int_0^1 k_1(u)\sin ut\,du\right\}dt$$

also belongs to L^2. Hence by (4.4.2) its Fourier transform is

$$-\frac{i\,\mathrm{sgn}\,x\,\mathcal{F}_1(x)}{1-\nabla_{k_1}}$$

where $$\nabla_{k_1} = \begin{cases} 1-\frac{1}{k_1(|x|)} & (|x|\le 1) \\ 0 & (|x|>1). \end{cases}$$

Hence the Fourier transform of

$$B_1'B_2'f$$

is

$$\left.\begin{array}{ll} -F(x)\,k_1(|x|)\,k_2(|x|) & (|x|\leq 1) \\ -F(x) & (|x|>1), \end{array}\right\}$$

and this is the same as the Fourier transform of $B_1 B_2 f$ because

$$h_1 h_2 = k_1 k_2 .$$

Hence the result in the special case.

We can now easily extend the case to get

$$B_1 B_2 \ldots . \; B_n f = B_1' B_2' \ldots . \; B_n' f$$

where $$h_1 h_2 \ldots \; h_n = k_1 k_2 \ldots \; k_n$$

and $k_\nu(x)$ is such that $\frac{1}{k_\nu(x)}$ satisfies both (3.1.3) and (3.4.2) and

$$\frac{J_\nu(x)}{x} \quad \varepsilon \quad L^2(-1, 1)$$

where J_ν, as mentioned above, is the Fourier transform of

$$B_{\nu+1}' B_{\nu+2}' \ldots \ldots \; B_n' f .$$

In like manner we may select more than one such B_ν' where $\frac{1}{k_\nu(x)}$ satisfies (3.1.3) & (3.4.2). But at each stage we have to see that the conditions of the theorem are satisfied. We demonstrate this by taking a number of particular cases.

To avoid confusion let us have a new notation to define the different kinds of the Boas transforms specially when

$$h(u) = u^n \qquad (n = 0,1,3,\ldots\ldots),$$

let us then write

$$f_n^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + \int_0^{\prime} u^n \sin ut \, du \right\} dt$$

for all positive odd integral values of n. We keep n odd to see that (3.1.3) are satisfied. Although , as we have already seen that

$$h(u) = u^n, \quad n = 0$$

does not satisfy strictly (3.1.3), yet this value could be taken because f_0^* exists almost everywhere and belongs to L^2. In fact

$$f_0^* = g.$$

When n = 1, we get the Goldberg transform and so it would be denoted now by f_1^*. Similarly $f_{1,1}^*$ would mean the Goldberg transform of the Goldberg transform of f (f^{**} of Theorem (G)). Similar explanation could be given for

$$f_{n,p}^* \qquad (n,p \text{ odd or zero})$$

which obviously exists almost everywhere and belongs to L^2. We of course know that

$$f_{n,p}^* = f_{p,n}^*.$$

We would rather prefer $f^{(2)*}$ and $f^{(3)*}$ to $f_{1,1}^*$ and $f_{1,1,1}^*$

for obvious reasons. Lastly by f_{-1}^{*} we would mean the <u>inverse</u> Goldberg transform of f, of course when it exists. For the general Boas transform we would use $f_{h}^{*}(x)$ and if the g.f. is

$$h^{n}(u) \qquad \text{(n odd or zero)},$$

the corresponding transform would be denoted by

$$f_{(n)}^{*}(x),$$

and then we may use $f_{(1)}^{*}(x)$ instead of $f_{h}^{*}(x)$ if we are not dealing simultaneously with h(x) and k(x) as g.fs.. It may be noted here that

$$f_{(0)}^{*} = f_{0}^{*} = g.$$

<u>Illustration 1.</u> We know from the Hilbert transform theory that

$$f_{0,0}^{*} = -f.$$

Now the kernel of the Compound transform $f_{0,0}^{*}$ is 1, for the g.fs. of the individual transforms involved are each 1, i.e.

$$h_{1}(x) = h_{2}(x) = 1.$$

Let us now select $\quad k_{1}(x) = \frac{1}{x} \quad \& \quad k_{2}(x) = x.$

Hence $$B_{2}' f = f_{1}^{*},$$

and if $\mathcal{F}_{1}$ be its Fourier transform then according to Heywood [6]

$$\frac{\mathcal{F}_{1}(x)}{x} \in L^{2}(-1, 1),$$

and so the conditions of the above theorem are satisfied, hence

$$f^*_{0,0} = B_1' B_2' f$$

where B_1' ($h(x) = 1/x$) stands for the inverse of the Goldberg transform and so

$$B_1' B_2' f = f^*_{1,-1} = -f.$$

Illustration 2. Again

$$f^*_{0,0,0,0} = f,$$

i.e. if we apply the Hilbert transform four times we get the original function.

Now let

$$k_1(x) = k_2(x) = k_3(x) = \frac{1}{x},$$

and

$$k_4(x) = x^3.$$

$k_4(x)$ satisfies (3.1.3) and $\frac{1}{k_1(x)}$, $\frac{1}{k_2(x)}$ and $\frac{1}{k_3(x)}$ satisfy both (3.1.3) & (3.4.2), and as

$$h_1(x) = h_2(x) = h_3(x) = h_4(x) = 1$$

so

$$h_1 h_2 h_3 h_4 = k_1 k_2 k_3 k_4.$$

Moreover

$$\frac{\mathcal{F}_3(x)}{x}, \quad \frac{M(x)}{x} \quad \text{and} \quad \frac{J(x)}{x}$$

all belong to $L^2(-1,1)$, where $\mathcal{F}_3(x)$, M(x) and J(x) are the Fourier transforms of f_3^*, $f^*_{3,-1}$ and $f^*_{3,-1,-1}$ respectively. Hence by Theorem (4.4)

$$f^{*}_{0,0,0,0} = f_3^{(-3)*} \qquad \left(f^{*}_{-1,-1,-1} \equiv f^{(-3)*} \right)$$

or $$f_3^{(-3)*} = f \qquad \ldots\ldots(4.4.3)$$

where, as in Illustration 1, we use f^{*}_{-1} for the inverse Goldberg transform of f(x) if it exists. In fact it exists in the case of f^{*}_{1}, for here

$$f^{*}_{1,-1} = -f$$

and so $$f^{(3)(-3)*} = -f.$$

Hence by (4.4.3)

$$f_3^{(-3)*} = -f^{(3)(-3)*}$$

or $$f_3^{*} = -f^{(3)*}. \qquad \ldots\ldots(4.4.4)$$

This shows that the kernel and the g.f. of $-f^{(3)*}$ are identical. It may be recalled here that a similar case was discussed in (4.2). Again by (3.9.2), when n = 2, the L.H.S. = $f^{(3)*}$ and the R.H.S. is

$$-g + 3g.o.\delta - 3g.o.\delta_2 + g.o.\delta_3$$

where δ and δ_k are as in (0.10) & (0.11) respectively. Hence by (4.4.4)

$$f_3^{*} = g - 3g.o.\delta + 3g.o\delta_2 - g.o.\delta_3 \qquad \ldots\ldots(4.4.5)$$

Illustration 3. The kernel of $f^{(2)*}$ is x^2, and so by the above theorem

$$f^{(2)*} = f^*_{3,-1},$$

since

$$x^2 = \frac{1}{x} \cdot x^3$$

i.e.

$$f^{(2)*} = \frac{1}{\pi} P \int_{-\infty}^{\infty} f_3^*(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 \frac{\sin ut}{u} du \right\} dt.$$

But by Theorem (G)

$$f^{(2)*} = -f + 2\delta.0.f - \delta.0.\delta.0.f.$$

Hence

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} f_3^*(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 \frac{\sin ut}{u} du \right\} dt = -f + 2\delta.0.f - \delta.0.\delta.0.f. \quad \ldots\ldots(4.4.6)$$

(4.4.3) and (4.4.6) may be considered as two methods of inverting f_3^*, or more precisely, expressing f(x) in terms of $f_3^*(x)$. A formula of the type (3.4.1) is not possible here, and for that matter, for any $f_n^*(x)$ (n > 1) because in these cases

$$\int_0^1 \frac{\sin ut}{u^n} du$$

does not exist.

The above two results can easily be generalised to cover the case of

$$f_n^*(x) \qquad \text{(n +ve odd integer)},$$

namely

$$f_n^{(-n)*} = (-1)^{\frac{n+1}{2}} f, \qquad \ldots\ldots(4.4.7)$$

and

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} f_n^*(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 \frac{\sin ut}{u}\, du \right\} dt =$$

$$= (-1)^{\frac{n-1}{2}} \left[f - (n-1)\delta.0.f + \cdots + \delta_{n-1}.0.f \right] \cdot \ldots\ldots(4.4.8)$$

(4.5) We now prove a general theorem on the equivalence of the kernel and the g.f., namely

THEOREM. Any Compound Boas transform of order '4n + 1' (n = 0,1,..) has its kernel and g.f. equal.

Proof. We first state a lemma.

LEMMA. If $$\Delta_{h_\nu} = \begin{matrix} 1 - h_\nu(|x|) & (|x| \le 1) \\ 0 & (|x| > 1) \end{matrix} \Bigg\}$$

and $$\Delta = \begin{matrix} 1 - \prod_{\nu=1}^{4n+1} h_\nu(|x|) & (|x| \le 1) \\ 0 & (|x| > 1), \end{matrix} \Bigg\}$$

then $$(-i\,\mathrm{sgn}\,x)^{4n+1} \prod_{\nu=1}^{4n+1} \{1 - \Delta_{h_\nu}\} = -i\,\mathrm{sgn}\,x \{1 - \Delta\}.$$

Let $B_1, B_2, \ldots\ldots, B_{4n+1}$ be 4n+1 Boas transforms with $h_1, h_2, \ldots\ldots, h_{4n+1}$ as their respective g.fs., then the kernel of

$$B_1 B_2 \ldots\ldots\ldots B_{4n+1} f \qquad \ldots\ldots(4.5.1)$$

is $h_1 h_2 \ldots\ldots\ldots h_{4n+1}$. The Fourier transform of (4.5.1) is obviously

$$(-i\,\mathrm{sgn}\,x)^{4n+1} F(x) \prod_{1}^{4n+1} \{1 - \Delta_{h_\nu}\}$$

$$= -i\,\mathrm{sgn}\,x\, F(x) \{1 - \Delta\} \qquad \text{(by the lemma).}$$

Hence taking the inverse Fourier transforms of each side, we have

$$B_1 B_2 \ldots\ldots B_{4n+1} f = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 \prod_{1}^{4n+1} h_\nu(u) \sin ut\, du \right\} dt.$$

CHAPTER V

THE DIFFERENTIAL OPERATOR h(D)

AND

THE BOAS TRANSFORM

(5.1) It has been shown in Chapter 2 that the Goldberg transform of a function of L^2 is the derivative of a function also of L^2. We now establish a similar result in connection with the Boas transform which has a polynomial as its generating function. As this result depends upon an extension of Theorem (M), we therefore prove the extension first.

Extension of Theorem (M). If ϕ and all its derivatives upto and including that of order n - 1 are absolutely continuous and

$$\phi, \phi', \phi'', \ldots\ldots\ldots, \phi^{(n)}$$

belong to L^2, then the Fourier transform of $\phi^{(n)}(x)$ is

$$(-ix)^n \Phi(x).$$

Conversely if $\Phi(x)$ and $x^n \Phi(x)$, (n +ve integer), belong to L^2 then all the derivatives of ϕ upto $\phi^{(n)}$ also belong to L^2.

Proof. The first part can easily be proved by the repeated application of Theorem (M). For the second part i.e. the converse we notice that when

$$x^n \Phi(x) \qquad \text{and} \qquad \Phi(x)$$

belong to L^2, then $x^{n-1} \Phi(x)$ also belongs to L^2, since it

is clear that

$$x^{n-1}\,\Phi(x) \;\varepsilon\; L^2(-\infty,-1) \text{ and } L^2(1,\infty)$$

for $n \geq 1$ ($F(x)/x \;\varepsilon\; L^2(-\infty,-1)$ & $L^2(1,\infty)$ if $F(x)\,\varepsilon\, L^2$).
Moreover as $\Phi(x)$ belongs to $L^2(-1,1)$

$$x^{p}\,\Phi(x) \qquad (p \geq 1)$$

belongs to $L^2(-1,1)$, and so

$$x^{n-1}\,\Phi(x)$$

belongs to L^2. Repeating the above argument we find that

$$x^{n-2}\,\Phi(x),\quad x^{n-3}\,\Phi(x),\quad \ldots\ldots,\quad x\,\Phi(x)$$

belong to L^2. The repeated application of the converse of Theorem (M) now gives the required result.

We also find here, by (0.13), that

$$\phi(x),\ \phi'(x),\ \phi''(x),\ldots\ldots,\ \phi^{(n-1)}(x)$$

tend to zero as $x \longrightarrow \pm\infty$. ....(5.1.1)

(5.2) <u>THEOREM</u>. If $f(x) \;\varepsilon\; L^2$, and $h(x)$ be a polynomial of degree $2n-1$ satisfying (3.1.3), and

$$f_h^*(x) = \frac{1}{\pi}\, P \int_{-\infty}^{\infty} f(x+t)\left\{\frac{\cos t}{t} + \int_0^{\prime} h(u)\sin ut\,du\right\}dt$$

then

$$f_h^*(x) = -i\,h(iD)\,\phi_1(x) + \phi_2'(x) \quad \ldots.(5.2.1)$$

where h(iD) is the differential operator obtained by replacing x in h(x) by

$$iD = i\frac{d}{dx},$$

and ϕ_1 and ϕ_2 are some functions belonging to L^2, such that

$$\left.\begin{aligned} \Phi_1(x) &= \begin{cases} F(x) & (|x|\le 1) \\ 0 & (|x|>1) \end{cases} \\ \Phi_2(x) &= \begin{cases} 0 & (|x|\le 1) \\ \dfrac{iG(x)}{x} & (|x|>1) \end{cases} \end{aligned}\right\} \quad \ldots.(5.2.2)$$

Φ_1, Φ_2, F, and G have the usual meaning.

Proof. As h(x) satisfies (3.1.3) it is, therefore, of the form

$$a_1 x^{2n-1} + a_2 x^{2n-3} + \ldots\ldots + a_n x$$

where the a_s are constants such that

$$\sum_{r=1}^{n} a_r = 1. \qquad \ldots.(5.2.3)$$

Thus $f_h^*(x)$ can be expressed as

$$\sum_{r=1}^{n} \frac{a_r}{\pi} P \int_{-\infty}^{\infty} f(x+t)\left\{\frac{\cos t}{t} + \int_0^1 u^{2n-2r+1}\sin ut\, du\right\} dt$$

$$= \sum_{r=1}^{n} a_r f_{2n-2r+1}^{*} \qquad \text{(in the notation of Ch. 4).}$$

$\ldots.(5.2.4)$

If $\mathcal{F}_\nu$ be the Fourier transform of f_ν^* then by (3.4.6)

$$F(x) = \begin{cases} \dfrac{i\,\mathcal{F}_\nu(x)}{x^\nu} & (|x|\le 1) \\ i\,\operatorname{sgn} x\,\mathcal{F}_\nu(x) & (|x|>1), \end{cases}$$

since here $h(x) = x^\nu$ (ν odd).

Hence

$$\frac{\mathcal{F}_\nu(x)}{x^\nu} \quad \varepsilon \quad L^2(-1, 1)$$

for all odd ν, and

$$\left.\begin{aligned} \frac{\mathcal{F}_\nu(x)}{x^\nu} &= \frac{\mathcal{F}_\mu(x)}{x^\mu} \quad (|x|\le 1) \\ \mathcal{F}_\nu(x) &= \mathcal{F}_\mu(x) \quad (|x|>1) \end{aligned}\right\} \ldots.(5.2.5)$$

for all odd values of μ and ν. As $\dfrac{\mathcal{F}_\nu(x)}{x^\nu} \varepsilon L^2(-\infty, -1)$ and $L^2(1, \infty)$ it belongs to L^2, let then

$$\left.\begin{aligned} \frac{i\,\mathcal{F}_\nu(x)}{x^\nu} &= \Phi_1(x) \quad (|x|\le 1) \\ &\chi(x) \quad (|x|>1) \end{aligned}\right\} \ldots.(5.2.6)$$

where $\Phi_1(x)$ and $\chi(x)$ vanish almost everywhere in $|x|>1$ and $|x|\le 1$ respectively. Obviously then

$$\Phi_1(x) = F(x) \qquad (|x|\le 1),$$

so that

$$\mathcal{F}_\nu(x) = -ix^\nu \Phi_1(x) \quad (|x|\le 1) \qquad \ldots.(5.2.7)$$

for all odd ν.

Now by (5.2.5) again for all odd ν

$$\mathcal{F}_\nu(x) = \mathcal{F}_1(x) \qquad (|x|>1).$$

Again as $\frac{\mathcal{F}_1(x)}{x} \in L^2$, let

$$\mathcal{F}_1(x) = -ix\,\Phi_2(x) \qquad (|x|>1)$$

where $\Phi_2(x)$ vanish almost everywhere in $|x| \leq 1$, then according to Goldberg [5]

$$\Phi_2 = \frac{i\,G(x)}{x}$$

since $\mathcal{F}_1(x) = G(x)$ in $|x|>1$.

Hence

$$\mathcal{F}_\nu(x) = -ix\,\Phi_2(x) \quad (|x|>1) \qquad \ldots.(5.2.8)$$

for all odd ν. Combining (5.2.7) and (5.2.8) we obtain for all odd ν

$$\mathcal{F}_\nu(x) = -ix^\nu \Phi_1(x) - ix\,\Phi_2(x).$$

Taking the inverse Fourier transforms of both sides and using the extension of Theorem (M) as proved in (5.1) we get for all positive odd integral values of ν

$$f_\nu^*(x) = -i(iD)^\nu \phi_1(x) + \phi_2'(x) \qquad \ldots.(5.2.9)$$

where, as usual, ϕ_1 and ϕ_2 are the inverse Fourier transforms of Φ_1 and Φ_2. In (5.2.6) we may also take

$$\Phi_1 + \chi = \Phi$$

then $\Phi = F$ in $|x| \leq 1$, and

$$\frac{i\,\mathcal{F}_\nu(x)}{x^\nu} = \Phi(x)$$

and so by (5.1)

$$f_\nu^*(x) = -(iD)^\nu \phi(x). \qquad \ldots\ldots(5.2.10)$$

Using now (5.2.4) and (5.2.9) we get

$$f_k^*(x) = -i\left[a_1(iD)^{2n-1} + a_2(iD)^{2n-3} + \ldots\ldots + a_n(iD)\right]\phi_1(x)$$

$$+ \sum_{r=1}^{n} a_r \phi_2'(x)$$

$$= -i\,h(iD)\,\phi_1(x) + \phi_2'(x). \quad \text{(by (5.2.3))}.$$

<u>Corollary 1.</u> All Boas transforms with polynomials as g.fs. can be expressed as derivatives of the first order of functions of L^2.

We need only prove here that

$$-\,i\,h(iD)\phi_1(x) = \psi'(x)$$

where $\psi(x) \in L^2$. Now

$$h(x) = x\left\{a_1 x^{2(n-1)} + \ldots\ldots + a_n\right\}$$

$$= x\,p(x), \text{ say,}$$

then

$$h(iD) = iD\,p(iD),$$

and as $\phi_1, \phi_1', \ldots\ldots\ldots, \phi_1^{(2n-1)}$ belong to L^2

$$\psi = p(iD)\phi_1$$

also belongs to L^2, hence

$$-ih(iD)\phi_1 = \psi.$$

Hence we may also say that $f_h^*(x)$ is, by Theorem (2.3), the Goldberg transform of an L^2 function.

Corollary 2. If $F(x) = 0$ (a.e) in $|x| > 1$, then $F(x) = \Phi_1(x)$ and so

$$f_h^*(x) = -ih(iD)f(x). \qquad \ldots.(5.2.11)$$

The above result is analogous to the theorem on the equivalence of the derived function and the Goldberg transform. Using Corollary 1, we may write (5.2.11) as

$$f_h^*(x) = \frac{d}{dx}\left(p(iD)\right)f(x).$$

On the other hand if $F(x) = 0$ (a.e) in $|x| \leq 1$, then

$$\Phi_1(x) = 0 \quad \text{(a.e)},$$

and so by (5.2.2) and (5.2.8) we get Theorem (D), namely

$$f_h^*(x) = g(x).$$

(5.3) The Function sin x/x.

We notice that the function sin x/x plays a very important part in the theory of the Boas transform including the particular case of the Goldberg transform. In fact the Goldberg transform formula

$$f_1^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t)\left\{\frac{\cos t}{t} + \int_0^1 u \sin ut\, du\right\} dt$$

may also be written as

$$f_1^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} - D\left(\frac{\sin t}{t}\right) \right\} dt$$

since $$D\left(\frac{\sin t}{t}\right) = D \int_0^1 \cos ut\, du = - \int_0^1 u \sin ut\, du,$$

D being the differential operator $\frac{d}{dt}$, and if

$$\frac{1}{D} \phi(t) = \int_0^t \phi(u)\, du \qquad \ldots.(5.3.1)$$

Heywood's inversion formula (0.12) may be written as

$$-f(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f_1^*(x+t) \left\{ \frac{\cos t}{t} + \frac{1}{D}\left(\frac{\sin t}{t}\right) \right\} dt.$$

Again as

$$\int_0^1 u^\nu \sin ut\, du = i(iD)^\nu \frac{\sin t}{t} = i(iD)^\nu \int_0^1 \cos ut\, du$$

for all positive odd integral values of ν, the Boas transform

$$f_\nu^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 u^\nu \sin ut\, du \right\} dt$$

may be written as

$$f_\nu^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + i(iD)^\nu \frac{\sin t}{t} \right\} dt. \qquad \ldots.(5.3.2)$$

Before we attempt to invert the formula (5.3.2) for the particular value $\nu = 3$, we prove a result involving

$$D^{-3}\left(\frac{\sin t}{t}\right),$$

namely

$$\int_{-\infty}^{\infty} f(x+t)\,\frac{\sin t}{t}\,dt = \lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)^2 f_3^*(x+t)\,D^{-3}\left(\frac{\sin t}{t}\right)dt \qquad \ldots\ldots(5.3.3)$$

which may also be written as

$$\int_{-\infty}^{\infty} f(x+t)\,\frac{\sin t}{t}\,dt = \lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)^2 f_3^*(x+t)\int_0^1 \frac{ut-\sin ut}{u^3}\,du\,dt,$$

since according to (5.3.1)

$$\frac{1}{D^3}\,\frac{\sin t}{t} = \int_0^1 \frac{ut-\sin ut}{u^3}\,du.$$

To prove this we need the following lemma.

LEMMA. If ϕ, ϕ' be absolutely continuous and ϕ, ϕ', and $\phi'' \in L^2$ then

(i) $\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}|\phi(x+t)|\,dt = 0$ for all x.

(ii) $\lim_{T\to\infty}\frac{1}{T^2}\int_{-T}^{T}|t\,\phi(x+t)|\,dt = 0$ " .

(iii) $$\lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} t\, \phi'(x+t)\,dt = 0 \qquad \text{for all } x.$$

(iv) $$\lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} \left(1-\frac{t}{T}\right) \phi'(x+t) \int_{0}^{t} \frac{1-\cos ut}{u^2}\,du\,dt = 0 \quad ,,$$

(v) $$\lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} \left(1-\frac{t}{T}\right) \phi''(x+t) \int_{0}^{t} \frac{ut-\sin ut}{u^3}\,du\,dt = 0 \quad ,,$$

Proof of the lemma. (i) and (ii) are proved by using Schwarz's inequality, since $\phi(t) \in L^2$ and $t \in L^2(-T, T)$. (ii) may also be deduced from (i) by using $|t| \leq T$.

(iii) is obtained on integrating by parts and then using (i).

(iv). Integrating by parts we obtain (+)

$$\lim_{T\to\infty} \frac{1}{T^2} \int_{0}^{T} \phi(x+t) \int_{0}^{t} \frac{1-\cos ut}{u^2}\,du\,dt -$$

$$- \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} \left(1-\frac{t}{T}\right) \phi(x+t) \int_{0}^{t} \frac{\sin ut}{u}\,du\,dt ,$$

the first term now tends to zero by (ii) and the second by (i), hence we get (iv).

(v). Here also on integrating by parts we get

$$\lim_{T\to\infty} \frac{1}{T^2} \int_{0}^{T} \phi'(x+t) \int_{0}^{t} \frac{ut-\sin ut}{u^3}\,du\,dt -$$

$$- \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} \left(1-\frac{t}{T}\right) \phi'(x+t) \int_{0}^{t} \frac{1-\cos ut}{u^2}\,du\,dt .$$

(+). The first term on integrating by parts becomes zero on substituting the limits. We omit it every time we integrate by parts under similar situations.

On further integrating by parts both the terms we get

$$\lim_{T\to\infty} \frac{1}{T^2}\left[\phi(x+t)\int_0^t \frac{ut-\sin ut}{u^3}\,du\right]_0^T -$$

$$-\lim_{T\to\infty}\frac{1}{T^2}\int_0^T \phi(x+t)\int_0^t \frac{1-\cos ut}{u^2}\,du\,dt -$$

$$-\lim_{T\to\infty}\frac{1}{T^2}\int_0^T \phi(x+t)\int_0^t \frac{1-\cos ut}{u^2}\,du\,dt$$

$$+\lim_{T\to\infty}\frac{1}{T}\int_0^T \left(1-\frac{t}{T}\right)\phi(x+t)\int_0^t \frac{\sin ut}{u}\,du\,dt.$$

Now the first term tends to 0, since $\phi(t)\to 0$ as $t\to\infty$, and

$$\left|\int_0^t \frac{ut-\sin ut}{u^3}\,du\right| \le Kt^2,$$

the second term tends to 0 by (ii) and so the third, the fourth tends to 0 by (i). Hence we obtain (v).

<u>Proof of (5.3.3)</u>. By (5.2.10) $f_3^*(x) = -\phi'''(x)$, where

$$F(x) = \Phi(x). \qquad (|x|\le 1).$$

Hence the L.H.S. of (5.3.3)

$$= \int_{-\infty}^{\infty} \phi(x+t)\frac{\sin t}{t}\,dt,$$

since by Theorem (L) the Fourier transforms of the L.H.S. of (5.3.3) and the above expression are equal.

We may also write

$$\int_{-\infty}^{\infty} \phi(x+t)\frac{\sin t}{t}dt = \lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)^2\phi(x+t)\frac{\sin t}{t}dt.$$

Integrating now the R.H.S. by parts we get

$$\lim_{T\to\infty}\frac{2}{T}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\operatorname{sgn} t\,\phi(x+t)\int_0^1\frac{\sin ut}{u}du\,dt -$$

$$-\lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)^2\phi'(x+t)\int_0^1\frac{\sin ut}{u}du\,dt.$$

The first term tends to 0 by (i) of Lemma, since

$$\int_0^1\frac{\sin ut}{u}du$$

is a bounded function of t, and the second term on integrating by parts gives

$$-\lim_{T\to\infty}\frac{2}{T}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\operatorname{sgn} t\,\phi'(x+t)\int_0^1\frac{1-\cos ut}{u^2}du\,dt +$$

$$+\lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)^2\phi''(x+t)\int_0^1\frac{1-\cos ut}{u^2}du\,dt.$$

The first term again tends to 0 by (iv) of Lemma, and the second term on further integration by parts becomes

$$\lim_{T\to\infty}\frac{2}{T}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\operatorname{sgn} t\,\phi''(x+t)\int_0^1\frac{ut-\sin ut}{u^3}du\,dt -$$

$$-\lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)^2\phi'''(x+t)\int_0^1\frac{ut-\sin ut}{u^3}\,du\,dt.$$

The first term tends to 0 by (v) of Lemma, and the second term, by (5.2.10) becomes

$$\lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)^2 f_3^*(x+t)\int_0^1\frac{ut-\sin ut}{u^3}\,du\,dt.$$

This completes the proof of (5.3.3).

We now prove a theorem on inversion.

(5.4) THEOREM. If $f(x)\in L^2$ and $f_3^*(x)$ be as given by (5.3.2) ($\nu=3$), then

$$-f(x)=\frac{1}{\pi}\lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)^2 f_3^*(x+t)\left\{\frac{\cos t}{t}+i(iD)^{-3}\frac{\sin t}{t}\right\}dt. \qquad \ldots.(5.4.1)$$

Proof. We may write

$$f(x)=-\frac{1}{\pi}P\int_{-\infty}^{\infty}f_3^*(x+t)\frac{\cos t}{t}\,dt+\frac{1}{\pi}\int_{-\infty}^{\infty}f(x+t)\frac{\sin t}{t}\,dt \qquad \ldots.(5.4.2)$$

since the Fourier transform of the first term on the right is by (3.2.2)

$$\left.\begin{array}{ll} 0 & (|x|\le 1)\\ i\,\mathrm{sgn}\,x\,\mathcal{F}_3(x) & (|x|>1)\end{array}\right\}$$

where, as usual, $\mathcal{F}_3$ is the Fourier transform of f_3^*, and by (3.4.6)

$$\mathcal{F}_3(x) = -i\,\mathrm{sgn}\,x\,F(x) \qquad (|x|>1).$$

Hence the Fourier transform of the first term of (5.4.2) is

$$\left.\begin{array}{ll} 0 & (|x|\le 1) \\ F(x) & (|x|>1) \end{array}\right\}$$

and that of the second term of (5.4.2) is by Theorem (L)

$$\left.\begin{array}{ll} F(x) & (|x|\le 1) \\ 0 & (|x|>1), \end{array}\right\}$$

(5.4.1) is now obtained by applying (5.3.3) to the second term of (5.4.2). Hence to sum up; if $f(x) \in L^2$ and

$$f_3^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + i(iD)^3 \frac{\sin t}{t} \right\} dt$$

then inversely

$$-f(x) = \frac{1}{\pi} \lim_{T\to\infty} \int_{-T}^{T} \left(1-\frac{|t|}{T}\right)^2 f_3^*(x+t) \left\{ \frac{\cos t}{t} + i(iD)^{-3} \frac{\sin t}{t} \right\} dt$$

Remark. We may prove in much the same way that

$$-f(x) = \frac{1}{\pi} \lim_{T\to\infty} \int_{-T}^{T} \left(1-\frac{|t|}{T}\right)^4 f_5^*(x+t) \left\{ \frac{\cos t}{t} + i(iD)^{-5} \frac{\sin t}{t} \right\} dt$$

where $f_5^*(x)$ is given by (5.3.2) ($\nu = 5$). Hence it appears that,

in general, for ν (ν any +ve odd integer) the formula for inversion may, most likely, be

$$-f(x) = \frac{1}{\pi} \lim_{T\to\infty} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right)^{\nu-1} f_{\nu}^{*}(x+t) \left\{ \frac{\cos t}{t} + i(iD)^{-\nu} \frac{\sin t}{t} \right\} dt.$$

(5.5) The above theory can also be applied to general Boas transform with polynomial as g.f., say as in (5.2)

$$h(x) = a_1 x^{2n-1} + a_2 x^{2n-3} + \ldots\ldots + a_n x$$

with $a_n \neq 0$, and such that h(x) has no other zero in (0,1).

In this case, if $f(x) \in L^2$ and $f_h^*(x)$ be its Boas transform with h(x) as given above, then

$$f_h^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + i h(iD) \frac{\sin t}{t} \right\} dt, \qquad \ldots\ldots(5.5.1)$$

and inversely

$$-f(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f_h^*(x+t) \left\{ \frac{\cos t}{t} + \frac{i}{h(iD)} \frac{\sin t}{t} \right\} dt. \qquad \ldots\ldots(5.5.2)$$

Proof. By the definition of $f_h^*(x)$

$$f_h^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 h(u) \sin ut \, du \right\} dt.$$

Now as $a_n \neq 0$, h(x) is of the form xp(x) where p(x) is an even polynomial (say = $q(x^2)$) and does not vanish anywhere in (0,1).

Hence h(x) satisfies conditions (3.4.2) and so by Theorem (3.4)

$$-f(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f_h^*(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 \frac{\sin ut}{h(u)} du \right\} dt.$$

Hence we need prove here that

$$ih(iD)\frac{\sin t}{t} = \int_0^1 h(u) \sin ut \, du, \quad \ldots\ldots(5.5.3)$$

and

$$\frac{i}{h(iD)} \frac{\sin t}{t} = \int_0^1 \frac{\sin ut}{h(u)} du. \quad \ldots\ldots(5.5.4)$$

Now as $h(x) = xp(x) = xq(x^2)$, $ih(iD) = -Dq(-D^2)$, and so

$$ih(iD)\frac{\sin t}{t} = -Dq(-D^2) \int_0^1 \cos ut \, du$$

$$= \int_0^1 q(-D^2) u \sin ut \, du = \int_0^1 h(u) \sin ut \, du,$$

since $q(-D^2)\sin ut = q(u^2)\sin ut$, and thus we get (5.5.3).

The case of (5.5.4) is similar except that here we have to justify the change of the order of integration. This of course is permissible here because $1/p(x)$ is bounded in $(0,1)$. In fact, here, we get

$$\frac{i}{h(iD)} \frac{\sin t}{t} = \frac{1}{q(-D^2)D} \frac{\sin t}{t} = \frac{1}{q(-D^2)} \int_0^1 \frac{\sin ut}{u} du$$

$$= \int_0^1 \frac{du}{u} \frac{1}{q(-D^2)} \sin ut$$

$$= \int_0^1 \frac{\sin ut}{h(u)} du,$$

and this completes the proof.

Note. The case $a_n = 0$ will be discussed in the next chapter, after we develop the theory of even generating functions. We will find there that the theory of even generating functions is a prerequisite for the discussion of a general odd polynomial.

CHAPTER VI

EVEN GENERATING FUNCTION

(6.1) Till now we have confined ourselves to the case when the generating function h(x) was an odd function. The way h(x) was introduced in our discussions of the Boas transform (cf. p.38) it could not be otherwise. But we notice that

$$\int_0^1 h(u) \sin ut \, du$$

exists and belongs to L^2, if h(u) is an even continuous function of bounded variation on (0,1) (one such example is that of h(u) = 1, when we get the Hilbert transform). We take first the simple case of $h(u) = u^\nu$ (ν any +ve even integer). We notice here that

$$(-1)^{\frac{\nu}{2}} \int_0^1 u^\nu \sin ut \, du$$

is the νth derivative of

$$\frac{1-\cos t}{t} = \int_0^1 \sin ut \, du .$$

Thus $$\sqrt{\frac{2}{\pi}} \, (iD)^\nu \frac{1-\cos t}{t} = \sqrt{\frac{2}{\pi}} \int_0^1 u^\nu \sin ut \, du \quad \ldots . (6.1.1)$$

The Fourier transform of

$$\sqrt{\frac{2}{\pi}} \, \frac{1-\cos t}{t}$$

is

$$\left.\begin{array}{ll} i \, \mathrm{sgn} x & (|x| \leq 1) \\ 0 & (|x| > 1). \end{array}\right\}$$

Hence by (5.1)

$$\sqrt{\frac{2}{\pi}}\ D^{\nu}\ \frac{1-\cos t}{t}\quad \varepsilon\quad L^2,$$

and its Fourier transform is

$$\left.\begin{array}{ll}(-ix)^{\nu}\, i\,\mathrm{sgn}\,x & (|x|\le 1)\\ 0 & (|x|>1).\end{array}\right\}$$

Hence that of (6.1.1) is

$$\left.\begin{array}{ll} i\,\mathrm{sgn}\,x\ x^{\nu} & (|x|\le 1)\\ 0 & (|x|>1).\end{array}\right\}$$

If now $f(x)\ \varepsilon\ L^2$,

$$-\frac{1}{\pi}\int_{-\infty}^{\infty} f(x+t)\int_0^1 u^{\nu}\sin ut\,du\,dt$$

also belongs to L^2, and its Fourier transform is by Theorem(L)

$$\left.\begin{array}{ll} i\,\mathrm{sgn}\,x\ x^{\nu}\,F(x) & (|x|\le 1)\\ 0 & (|x|>1).\end{array}\right\}\ldots.(6.1.2)$$

By (3.2.2) the Fourier transform of

$$\frac{1}{\pi}\,P\int_{-\infty}^{\infty} f(x+t)\,\frac{\cos t}{t}\,dt$$

is

$$\left.\begin{array}{ll} 0 & (|x|\le 1)\\ -\,i\ \mathrm{sgn}x F(x) & (|x|>1).\end{array}\right\}$$

Hence for even ν

$$\frac{1}{\pi}\,P\int_{-\infty}^{\infty} f(x+t)\left\{\frac{\cos t}{t}+\int_0^1 u^{\nu}\sin ut\,du\right\}dt$$

$= f_\nu^*(x)$, say, exists for almost all x and belongs to L^2, and if, as usual, $\mathcal{F}_\nu$ be its Fourier transform then

$$\mathcal{F}_\nu(x) = -i\,\mathrm{sgn}\,x\, F(x)\{1-\Delta_\nu\},$$

where for even ν

$$\Delta_\nu = \begin{cases} 1-x^\nu & (|x|\le 1) \\ 0 & (|x|>1) \end{cases}$$

or

$$F(x) = \begin{cases} \dfrac{i\,\mathrm{sgn}\,x\,\mathcal{F}_\nu(x)}{x^\nu} & (|x|\le 1) \\ i\,\mathrm{sgn}\,x\,\mathcal{F}_\nu(x) & (|x|>1). \end{cases}$$

Hence for even ν $\dfrac{\mathcal{F}_\nu(x)}{x^\nu} \in L^2$.

Let now $\phi(x) \in L^2$, and be such that

$$\Phi(x) = \frac{\mathcal{F}_\nu(x)}{x^\nu} \quad \text{or} \quad \mathcal{F}_\nu(x) = x^\nu \Phi(x),$$

i.e. $$\Phi(x) = G(x) \qquad (|x|\le 1).$$

Hence by (5.1)

$$f_\nu^*(x) = (iD)^\nu \phi(x) \qquad \ldots\ldots(6.1.3)$$

which shows that even here, as in (5.2.10), $f_\nu^*(x)$ is the ν th derivative of a function of L^2.

If now $F(x) = 0 \quad (|x|>1)$

then $G(x) = 0$ "

and so $\phi(x) = g(x)$.

Hence for even ν

$$f_\nu^*(x) = (i\,D)^\nu g(x). \qquad \ldots.(6.1.4)$$

We may recall here that when ν is odd, then under the above circumstance (cf. (5.2.11))

$$f_\nu^*(x) = -i\,(i\,D)^\nu f(x).$$

The formula (6.1.3) can also be expressed in a form similar to (5.2.1). Let ϕ_1 and ϕ_2 be two L^2 functions such that

$$\Phi_1 = \begin{cases} G & (|x| \le 1) \\ 0 & (|x| > 1), \end{cases}$$

and

$$\Phi_2 = \begin{cases} 0 & (|x| \le 1) \\ G & (|x| > 1), \end{cases}$$

then for $|x| > 1$ $\quad \mathcal{F}_0 = G = \Phi_2$

and as for $|x| > 1$, and ν even $\quad \mathcal{F}_\nu = \mathcal{F}_0$

we get for $|x| > 1$, and ν even $\quad \mathcal{F}_\nu = \Phi_2$.

Moreover for $|x| \le 1$ $\quad \mathcal{F}_\nu = x^\nu \Phi_1$.

Hence for ν even $\quad \mathcal{F}_\nu = x^\nu \Phi_1 + \Phi_2$,

and so by (5.1), when ν is even

$$f_\nu^* = (i\,D)^\nu \phi_1 + \phi_2. \qquad \ldots.(6.1.5)$$

We can now prove theorems similar to (5.2) and (5.4).

(6.2) THEOREM. Let $f(x) \in L^2$, and $h(x)$ be an even polynomial of degree $2n$, such that $h(0) = 0$, $h(1) = 1$, and $f_h^*(x)$ be the Boas transform referred to the above $h(x)$ as its generating function, i.e.

$$f_h^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 h(u) \sin ut \, du \right\} dt$$

then $$f_h^*(x) = h(iD)\phi_1(x) + \phi_2(x),$$

where $$\Phi_1 = \begin{cases} G & (|x| \leq 1) \\ 0 & (|x| > 1) \end{cases}$$

and $$\Phi_2 = \begin{cases} 0 & (|x| \leq 1) \\ G & (|x| > 1), \end{cases}$$

i.e. ϕ_1 and ϕ_2 are the functions of (6.1.5).

Proof. As in (5.2)

$$f_h^*(x) = \sum_{r=1}^{n} \frac{a_r}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 u^{2n-2r+2} \sin ut \, du \right\} dt$$

$$= \sum_{r=1}^{n} a_r \, f_{2n-2r+2}^*(x)$$

$$= \sum_{r=1}^{n} a_r \left[(iD)^{2n-2r+2} \phi_1(x) + \phi_2(x) \right] \text{ (by (6.1.5))}$$

$$= h(iD)\phi_1(x) + \phi_2(x),$$

since here also $\sum_{r=1}^{n} a_r = 1$, and of course $h(x) = \sum_{r=1}^{n} a_r x^{2n-2r+2}$.

Corollary. If $F(x) = 0$ (a.e) for $|x| > 1$, we get $\phi_2(x) = 0$ (a.e), since $\Phi_2 = G = 0$ for $|x| > 1$, and $\Phi_1 = G$ (a.e), and so when $h(x)$ is even

$$f_h^*(x) = h(iD)g(x). \qquad \ldots.(6.2.1)$$

It may be noted that (6.1.4) is a particular case of (6.2.1).

In case of $F(x) = 0$ (a.e) for $|x| \leq 1$, we find that

$$\phi_1(x) = 0 \quad \text{(a.e)}$$

since $\Phi_1 = G = 0$ for $|x| \leq 1$,

and $\Phi_2 = G$ (a.e).

Hence $\phi_2 = g$,

and so

$$f_h^*(x) = g(x). \qquad \ldots.(6.2.2)$$

This shows that Theorem (D) is true even when the g.f. is even.

(6.3) THEOREM. If $f(x)$ and $f_\nu^*(x)$ be as above and ν an even positive integer, then

$$f_\nu^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t)\left[\frac{\cos t}{t} + (iD)^\nu \frac{1-\cos t}{t}\right] dt, \qquad \ldots.(6.3.1)$$

and inversely

$$-f(x) = \frac{1}{\pi} \lim_{T\to\infty} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right)^2 f_2^*(x+t)\left[\frac{\cos t}{t} + (iD)^{-2} \frac{1-\cos t}{t}\right] dt. \qquad \ldots.(6.3.2)$$

We first prove a lemma similar to the one we proved in (5.3).

LEMMA. If $\phi \in L^2$, then

(i) $\lim_{T\to\infty} \frac{1}{T}\int_1^T \log t\,|\phi(x\pm t)|\,dt = 0$ for all x.

(ii) $\lim_{T\to\infty} \frac{1}{T}\int_0^T \phi(x\pm t)\int_0^t \frac{1-\cos ut}{u}\,du\,dt = 0$.

(iii) $\lim_{T\to\infty} \frac{1}{T^2}\int_0^T \phi(x\pm t)\int_0^t \frac{ut-\sin ut}{u^2}\,du\,dt = 0$.

Proof of the lemma.

(i) Using Schwarz's inequality we get

$$\left|\int_1^T \log t\,|\phi(x\pm t)|\,dt\right| \le \sqrt{\int_1^T \log^2 t\,dt \int_{-\infty}^{\infty} \{\phi(x\pm t)\}^2 dt}$$

$$\le K\sqrt{\int_1^T \log^2 t\,dt} = K\left\{T\log^2 T - 2T\log T + 2T - 2\right\}^{\frac{1}{2}}.$$

Hence

$$\lim_{T\to\infty} \frac{1}{T}\left|\int_1^T \log t\,|\phi(x\pm t)|\,dt\right|$$

$$\le K\lim_{T\to\infty}\left\{\frac{\log^2 T}{T} - \frac{2\log T}{T} + \frac{2}{T} - \frac{2}{T^2}\right\}^{\frac{1}{2}}$$

$$= 0.$$

(ii) $$\int_0^T \phi(x\pm t)\int_0^t \frac{1-\cos ut}{u}\,du\,dt = \int_0^T \phi(x\pm t)\int_0^t \frac{1-\cos u}{u}\,du\,dt$$

$$= \left[\int_0^1 + \int_1^T\right]\phi(x\pm t)\int_0^t \frac{1-\cos u}{u}\,du\,dt.$$

Now by (i) of lemma of (5.3)

$$\lim_{T\to\infty} \frac{1}{T}\int_0^1 \phi(x\pm t)\int_0^t \frac{1-\cos u}{u}\,du\,dt = 0,$$

and

$$\frac{1}{T}\int_1^T \phi(x\pm t)\int_0^t \frac{1-\cos u}{u}\,du\,dt = \frac{1}{T}\int_1^T \phi(x\pm t)\left\{\int_0^1 + \int_1^t\right\}\frac{1-\cos u}{u}\,du\,dt.$$

Again by (i) of lemma of (5.3)

$$\lim_{T\to\infty}\frac{1}{T}\int_1^T \phi(x\pm t)\int_0^1 \frac{1-\cos u}{u}\,du\,dt = 0,$$

since $t \geqslant 1$, and as

$$\left|\int_1^t \frac{1-\cos u}{u}\,du\right| \leq 2\int_1^t \frac{du}{u} = 2\log t,$$

$$\frac{1}{T}\left|\int_1^T \phi(x\pm t)\int_1^t \frac{1-\cos u}{u}\,du\,dt\right| \leqslant \frac{2}{T}\int_1^T \log t\,|\phi(x\pm t)|\,dt,$$

and so tends to 0 as $T\to\infty$ by (i). Thus we get (ii).

(iii)
$$\lim_{T\to\infty}\frac{1}{T^2}\int_0^1 \phi(x\pm t)\int_0^t \frac{u-\sin u}{u^2}\,du\,dt = 0$$

by (i) of lemma of (5.3), since

$$\int_0^t \frac{u-\sin u}{u^2}\,du$$

is bounded in $0 \leqslant t \leqslant 1$.

Now for $t > 1$

$$\int_0^1 \frac{ut-\sin ut}{u^2}\,du = t\int_0^t \frac{u-\sin u}{u^2}\,du$$

$$\leqslant t\left[\int_0^1 + \int_1^t\right]\frac{u-\sin u}{u^2}\,du$$

$$< Kt + 2t\log t \qquad (K \text{ a constant}),$$

Hence

$$\frac{1}{T^2}\left|\int_1^T \phi(x\pm t)\int_0^t \frac{ut-\sin ut}{u^2}\,du\,dt\right|$$

$$\leq \frac{K}{T^2}\int_1^T t\,|\phi(x\pm t)|\,dt$$

$$+\frac{2}{T^2}\int_1^T t\log t\,|\phi(x\pm t)|\,dt,$$

and both these tend to 0 as $T\to\infty$, the first by (ii) of lemma of (5.3) and the second by (i) above. This completes the proof of (iii).

Proof of the theorem. (6.3.1) follows at once from (6.1.1). For (6.3.2) we write, as in (5.4.2)

$$-f(x)=\frac{1}{\pi}P\int_{-\infty}^{\infty}f_2^*(x+t)\frac{\cos t}{t}-\frac{1}{\pi}\int_{-\infty}^{\infty}f(x+t)\frac{\sin t}{t}\,dt. \qquad \ldots.(6.3.3)$$

Now the second term (with the -ve sign) on the right of (6.3.3) is by Theorem (B)

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}g(x+t)\frac{1-\cos t}{t}\,dt$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\phi(x+t)\frac{1-\cos t}{t}\,dt \qquad \ldots.(6.3.4)$$

where $\phi(x)$ is as in (6.1.3), so that for $|x|\leq 1$

$$\Phi(x)=G(x).$$

We may also write (6.3.4) as

$$\lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right)^2 \phi(x+t) \frac{1-\cos t}{t} dt .$$

Integrating it by parts and, as usual, discarding the first term (it becomes zero on substituting the limits), we get

$$\lim_{T\to\infty} \frac{2}{\pi T} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right) \operatorname{sgn} t \, \phi(x+t) \int_0^t \frac{1-\cos u}{u} du dt -$$

$$- \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right)^2 \phi'(x+t) \int_0^t \frac{1-\cos u}{u} du dt ,$$

the first term tends to 0 by (ii) of lemma and the second on further integrating by parts becomes

$$- \lim_{T\to\infty} \frac{2}{\pi T} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right) \operatorname{sgn} t \, \phi'(x+t) \int_0^t \frac{ut - \sin ut}{u^2} du dt +$$

$$+ \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right)^2 \phi''(x+t) \int_0^t \frac{ut - \sin ut}{u^2} du dt .$$

If we again integrate by parts the first term we obtain

$$- \lim_{T\to\infty} \frac{2}{\pi T^2} \int_{-T}^{T} \phi(x+t) \int_0^t \frac{ut - \sin ut}{u^2} du dt +$$

$$+ \lim_{T\to\infty} \frac{2}{\pi T} \int_{-T}^{T} \phi(x+t) \left(1 - \frac{|t|}{T}\right) \operatorname{sgn} t \int_0^t \frac{1-\cos ut}{u} du dt ,$$

and both these tend to 0 by (iii) and (ii) of lemma respectively. Hence we find that (6.3.4)

$$= \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right)^2 \phi''(x+t) \int_0^t \frac{ut - \sin ut}{u^2} du dt$$

$$= -\lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \left(1-\frac{|t|}{T}\right)^2 f_2^*(x+t) \int_0^1 \frac{ut-\sin ut}{u^2}\, du\, dt,$$

since by (6.1.3) $f_2^*(x) = -\phi''(x)$.

Substituting this value in (6.3.3) we get (6.3.2), since by the definition of the inverse operator 1/D, as given in (5.3.1), we find that

$$\int_0^1 \frac{ut-\sin ut}{u^2}\, du = \frac{1}{D^2}\,\frac{1-\cos t}{t}.$$

We notice that the formula (6.3.1) is true for all positive even integral values of ν, but the inverse formula (6.3.2) is proved only for $\nu = 2$. We can, in like manner, prove it for $\nu = 4$, and so it appears that the general inversion formula for ν, an even positive integer, could possibly be

$$-f(x) = \frac{1}{\pi} \lim_{T\to\infty} \int_{-T}^{T} \left(1-\frac{|t|}{T}\right)^{\nu} f_{\nu}^*(x+t) \left[\frac{\cos t}{t} + (i\cdot D)^{\nu}\frac{1-\cos t}{t}\right] dt.$$

(6.4) THEOREM. If $f(x) \in L^2$, and $h(x)$ be an even continuous function of bounded variation on $(0,1)$ then

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 h(u) \sin ut\, du \right\} dt \qquad \ldots\ldots(6.4.1)$$

exists for almost all x and belongs to L^2.

Proof. Consider

$$\Psi(x) = \begin{matrix} i\,\mathrm{sgn}\,x\,h(x) & (|x|\le 1) \\ 0 & (|x|>1) \end{matrix} \Big\}$$

then $\Psi(x) \in L^2$. Let $\psi(x)$ be its inverse Fourier transform, then

$$\psi(x) = \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} \mathrm{sgn}\,t\,h(t)\,e^{-ixt}\,dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 h(t) \sin xt\,dt .$$

Hence the second term of (6.4.1) is the convolution of $f(x)$ and $\psi(-x)$, and so by Theorem (L) has

$$\begin{matrix} -i\,\mathrm{sgn}\,x\,h(x)F(x) & (|x|\le 1) \\ 0 & (|x|>1) \end{matrix} \Big\} \quad \ldots\ldots(6.4.2)$$

as its Fourier transform. Again by Theorem (3.2) the first term of (6.4.1) belongs to L^2. Hence (6.4.1) exists for almost all x as an L^2 function, say as usual, $f_h^*(x)$. If $\mathcal{F}$ be its Fourier transform, then by (3.2.2) and (6.4.2)

$$\mathcal{F}(x) = -i\,\mathrm{sgn}\,x\,F(x)\{1-\Delta_h\} \quad \ldots\ldots(6.4.3)$$

where

$$\Delta_h = \begin{matrix} 1-h(x) & (|x|\le 1) \\ 0 & (|x|>1). \end{matrix} \Big\}$$

(6.5) In the light of the theory developed above we see that theorems proved earlier in connection with Boas transforms with odd g.fs. such as (3.7) and (4.3) are also true

here. The proofs will exactly be similar to those of (3.7) and (4.3). In case of (4.3) we may have a general theorem covering both odd and even values of h(x), viz

GENERAL THEOREM. All compound Boas transforms of the same order and kernel are equivalent almost everywhere, where the individual transforms are not necessarily generated either by all odd h(x) or by all even h(x).

Proof. Let, as usual, the order be 2. Let $h_1(x)$ and $k_1(x)$ be even and $h_2(x)$ and $k_2(x)$ be odd, and let B_1, B_2, B_1', B_2' be the Boas transforms generated by h_1, h_2, k_1, k_2 respectively. Let

$$h_1(x)\, h_2(x) \;=\; k_1(x)\, k_2(x),$$

then we have to show that

$$B_1 B_2 f \;=\; B_1' B_2' f.$$

Now as h_1 and k_1 are even

$$h_1(x)\, h_2(|x|) = k_1(x)\, k_2(|x|) \qquad \ldots.(6.5.1)$$

The Fourier transform of $B_1 B_2 f$ is

$$\left.\begin{array}{ll} -F(x)\, h_1(x)\, h_2(|x|) & (|x| \le 1) \\ -F(x) & (|x| > 1) \end{array}\right\}$$

$$= \left.\begin{array}{ll} -F(x)\, k_1(x)\, k_2(|x|) & (|x| \le 1) \\ -F(x) & (|x| > 1) \end{array}\right\} \text{(by (6.5.1))}$$

and this is the same as the Fourier transform of $B_1'B_2'f$. Hence the result.

Remark. It may be noted that we have already used an even g.f., viz $h(x) = 1$ in the illustrations given in (4.4).

(6.6) THEOREM. If $h(x)$ be as defined in (6.4) and be such that

$$h(x) = 0 \quad \text{nowhere in } (0,1),$$

and if

$$f_h^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + \int_0' h(u) \sin ut \, du \right\} dt$$

then

$$-f(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f_h^*(x+t) \left\{ \frac{\cos t}{t} + \int_0' \frac{\sin ut}{h(u)} du \right\} dt. \qquad \ldots\ldots(6.6.1)$$

Proof. (First Method). As $h(x)$ vanishes nowhere in $(0,1)$ $1/h(x)$ satisfies the conditions of Theorem (6.4), and as such

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \phi(x+t) \left\{ \frac{\cos t}{t} + \int_0' \frac{\sin ut}{h(u)} du \right\} dt$$

exists for almost all x as an L^2 function, provided $\phi \in L^2$, and may be denoted, according to our notations, by $\phi_{\frac{1}{h}}^*(x)$. Now by (6.4)

$$f_h^*(x) \in L^2.$$

Hence the R.H.S. of (6.6.1)

$$= \left[f_h^* \right]^*_{\frac{1}{h}} = f^*_{h,\frac{1}{h}}.$$

Now $f^*_{h,\frac{1}{h}}$ is a compound Boas transform of order 2 and kernel $h/h = 1$. Hence by Theorem (6.5), if H(f) denotes the Hilbert transform of f,

$$f^*_{h,\frac{1}{h}} = H^2(f),$$

and it is known that $H^2(f) = -f$. Hence the required result.

(Second Method). As h(x) does not vanish anywhere in (0,1), and if

$$\Psi(x) = \left.\begin{array}{ll} \dfrac{i \operatorname{sgn} x}{h(x)} & (|x| \leq 1) \\ 0 & (|x| > 1) \end{array}\right\} \quad \ldots\ldots(6.6.2)$$

then $\Psi(x) \in L^2$, and if $\psi(x)$ be its inverse Fourier transform then

$$\psi(x) = \sqrt{\frac{2}{\pi}} \int_0^1 \frac{\sin xt}{h(t)}\, dt.$$

Hence the Fourier transform of

$$-\frac{1}{\pi} \int_{-\infty}^{\infty} f_h^*(x+t) \int_0^1 \frac{\sin ut}{h(u)}\, du\, dt$$

is by Theorem (L) $\mathcal{F}(x)\,\Psi(x)$, where $\mathcal{F}(x)$ is the Fourier

transform of $f_h^*(x)$, and therefore by (3.2.2) the Fourier transform of the R.H.S. of (6.6.1) is

$$-\frac{i\,\mathrm{sgn}\,x\;\mathcal{F}(x)}{1-\Delta_h}$$

where Δ_h is the same as in (6.4), and this by (6.4.3) is-F(x). Hence the required result.

(6.7) We now consider the case when h(x) is an even polynomial, say

$$h(x) = a_1 x^{2n} + a_2 x^{2n-2} + \cdots\cdots + a_n x^2 + a_{n+1}.$$

Here h(x) satisfies the conditions of Theorem (6.4), hence

$$f_h^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 h(u) \sin ut\, du \right\} dt \qquad \ldots\ldots(6.7.1)$$

exists for almost all x, and belongs to L^2. For inversion we consider the different cases separately.

Case I. h(x) $\neq$ 0 anywhere in (0,1).

Hence it satisfies the conditions of Theorem (6.6), and so

$$-f(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f_h^*(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 \frac{\sin ut}{h(u)}\, du \right\} dt. \qquad \ldots\ldots(6.7.2)$$

Using the differential operator h(D) the two formulae (6.7.1) and (6.7.2) can also be written as

$$f_h^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + h(iD) \frac{1-\cos t}{t} \right\} dt, \quad \ldots\ldots(6.7.3)$$

and

$$-f(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f_h^*(x+t) \left\{ \frac{\cos t}{t} + \frac{1}{h(iD)} \frac{1-\cos t}{t} \right\} dt \ldots\ldots(6.7.4)$$

Proof. As h(x) is even, let it be $p(x^2)$, then

$$h(iD) \frac{1-\cos t}{t} = h(iD) \int_0^1 \sin ut \, du$$

$$= \int_0^1 p(-D^2) \sin ut \, du = \int_0^1 h(u) \sin ut \, du,$$

and

$$\frac{1}{h(iD)} \frac{1-\cos t}{t} = \frac{1}{p(-D^2)} \int_0^1 \sin ut \, du$$

$$= \int_0^1 \frac{\sin ut}{h(u)} \, du,$$

since $h(x) \neq 0$ in (0,1).

Case II. $h(x) = 0$ only at $x = 0$, and nowhere else, i.e.

$$a_{n+1} = 0.$$

Hence $$h(x) = x^2 \left[a_1 x^{2n-2} + \ldots\ldots + a_n \right]$$

$$= x^2 p(x), \text{ say,}$$

where $p(x)$ is an even polynomial such that

$$p(x) \neq 0 \text{ anywhere in } (0,1),$$

i.e. $p(x)$ satisfies the conditions of (6.6), and so

$$\sqrt{\frac{2}{\pi}} \int_0^1 \frac{\sin ut}{p(u)}\, du$$

is a bounded function of t, and belongs to L^2, with

$$\left.\begin{array}{ll} \dfrac{i\,\mathrm{sgn}\,x}{p(x)} & (|x| \le 1) \\ 0 & (|x| > 1) \end{array}\right\} \qquad \ldots.(6.7.5)$$

as its Fourier transform. Now by (6.4.3)

$$F(x) = \left.\begin{array}{ll} \dfrac{i\,\mathrm{sgn}\,x\ \mathcal{F}(x)}{x^2 p(x)} & (|x| \le 1) \\ i\,\mathrm{sgn}\,x\ \mathcal{F}(x) & (|x| > 1) \end{array}\right\} \qquad \ldots.(6.7.6)$$

and so $\dfrac{\mathcal{F}(x)}{x^2} \in L^2(-1,1)$ or for that matter to L^2. If we write

$$\underline{\Phi}(x) = \frac{\mathcal{F}(x)}{x^2}$$

or

$$\mathcal{F}(x) = x^2\, \underline{\Phi}(x) \qquad \ldots.(6.7.7)$$

then both $\underline{\Phi}(x)$ and $x^2\, \underline{\Phi}(x)$ belong to L^2, and thus by (5.1) ϕ (the inverse Fourier transform of $\underline{\Phi}$), ϕ' and ϕ'' all belong to L^2. Hence taking the inverse Fourier transforms of (6.7.7) we get

$$f_h^*(x) = -\phi''(x). \qquad \ldots.(6.7.8)$$

Now as in (6.3.3)

$$f(x) = -\frac{1}{\pi} P\int_{-\infty}^{\infty} f_h^*(x+t)\frac{\cos t}{t} + \frac{1}{\pi}\int_{-\infty}^{\infty} f(x+t)\frac{\sin t}{t}\,dt. \qquad \ldots.(6.7.9)$$

As before the Fourier transform of the second term on the right is given by

$$\left.\begin{matrix} F(x) & (|x|\le 1) \\ 0 & (|x|>1) \end{matrix}\right\}$$

$$= \left.\begin{matrix} \frac{i\,\mathrm{sgn}\,x}{p(x)}\,\bar{\Phi}(x) & (|x|\le 1) \\ 0 & (|x|>1), \end{matrix}\right\} \text{(by (6.7.6)\&(6.7.7))}$$

and the inverse Fourier transform of this is

$$-\frac{1}{\pi}\int_{-\infty}^{\infty}\phi(x+t)\int_0^1 \frac{\sin ut}{p(u)}\,du\,dt. \qquad \text{(by (6.7.5))}$$

Hence

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f(x+t)\frac{\sin t}{t}\,dt = -\frac{1}{\pi}\int_{-\infty}^{\infty}\phi(x+t)\int_0^1 \frac{\sin ut}{p(u)}\,du\,dt. \qquad \ldots.(6.7.10)$$

Before we proceed further we prove a lemma similar to the one we have proved in (6.3).

LEMMA. If $\phi \in L^2$, then

(i) $$\lim_{T\to\infty}\frac{1}{T}\int_1^T \phi(x\pm t)\int_0^1 \frac{1-\cos ut}{u\,p(u)}\,du\,dt = 0.$$

(ii) $$\lim_{T\to\infty} \frac{1}{T^2} \int_1^T \phi(x\pm t) \int_0^1 \frac{ut - \sin ut}{u^2 p(u)}\, du\, dt = 0.$$

Proof of the lemma.

(i)

$$\frac{1}{T}\int_1^T \phi(x\pm t)\int_0^1 \frac{1-\cos ut}{up(u)}\, du\, dt = \frac{1}{T}\int_1^T \phi(x\pm t)\int_0^t \frac{1-\cos u}{up(\frac{u}{t})}\, du\, dt$$

$$= \frac{1}{T}\int_1^T \phi(x\pm t)\int_0^1 \frac{1-\cos u}{up(\frac{u}{t})}\, du\, dt + \frac{1}{T}\int_1^T \phi(x\pm t)\int_1^t \frac{1-\cos u}{up(\frac{u}{t})}\, du\, dt.$$

The first term tends to 0 as $T \to \infty$, since $1/p(u/t)$, for $t > 1$, is bounded ($p(u)$ is bounded and non-zero in $(0,1)$). In the second term, for $t > 1$

$$\left| \int_1^t \frac{1-\cos u}{up(\frac{u}{t})}\, du \right| \leq K \int_1^t \frac{1-\cos u}{u}\, du \quad (K \text{ a constant})$$

$$\leq 2K \int_1^t \frac{du}{u} = 2K \log t.$$

Thus

$$\frac{1}{T}\left| \int_1^T \phi(x\pm t)\int_1^t \frac{1-\cos u}{up(\frac{u}{t})}\, du\, dt \right|$$

$$\leq \frac{2K}{T}\int_1^T \log t\, |\phi(x\pm t)|\, dt,$$

and so $\to 0$ as $T \to \infty$, by (i) of lemma of (6.3).

(ii) As in (i)

$$\frac{1}{T^2}\int_1^T \phi(x\pm t)\int_0^1 \frac{ut-\sin ut}{u^2 p(u)}\,du\,dt = \frac{1}{T^2}\int_1^T t\,\phi(x\pm t)\int_0^t \frac{u-\sin u}{u^2 p(\frac{u}{t})}\,du\,dt$$

$$= \frac{1}{T^2}\int_1^T t\,\phi(x\pm t)\int_0^1 \frac{u-\sin u}{u^2 p(\frac{u}{t})}\,du\,dt +$$

$$+ \frac{1}{T^2}\int_1^T t\,\phi(x\pm t)\int_1^t \frac{u-\sin u}{u^2 p(\frac{u}{t})}\,du\,dt.$$

The first term tends to 0 as $T \longrightarrow \infty$, by (ii) of (5.3) since

$$\int_0^1 \frac{u-\sin u}{u^2 p(\frac{u}{t})}\,du$$

is bounded for $t \geq 1$, and in the second term

$$\int_1^t \left|\frac{u-\sin u}{u^2 p(\frac{u}{t})}\right| du \leq 2M\int_1^t \frac{du}{u} = 2M\log t$$

(M a constant),

and so the second term also tends to 0 as $T \longrightarrow \infty$, by the lemma of (6.3). This completes the proof of the lemma.

It may be noted here that

$$\lim_{T\to\infty} \frac{1}{T^2}\int_0^1 \phi(x\pm t)\int_0^1 \frac{ut-\sin ut}{u^2 p(u)}\,du\,dt = 0.$$

<u>Proof of the ~~theorem~~ (Case II)</u>. Let us write the R.H.S. of (6.7.10) as

$$-\lim_{T\to\infty} \frac{1}{\pi}\int_{-T}^T \left(1-\frac{|t|}{T}\right)^2 \phi(x+t)\int_0^1 \frac{\sin ut}{p(u)}\,du\,dt.$$

Integrating it by parts and discarding the first term as it vanishes on substituting the limits we get

$$-\lim_{T\to\infty}\frac{2}{\pi T}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\operatorname{sgn}t\,\phi(x+t)\int_0^1\frac{1-\cos ut}{up(u)}\,du\,dt+$$

$$+\lim_{T\to\infty}\frac{1}{\pi}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)^2\phi'(x+t)\int_0^1\frac{1-\cos ut}{up(u)}\,du\,dt.$$

The first term tends to 0 by (i) of lemma above, since here

$$\lim_{T\to\infty}\frac{1}{T}\int_{-1}^{1}\operatorname{sgn}t\,\phi(x+t)\int_0^1\frac{1-\cos ut}{up(u)}\,du\,dt$$

is obviously zero because $-1\le t\le 1$.

Integrating further the second term and, as usual, discarding the first term we get

$$\lim_{T\to\infty}\frac{2}{\pi T}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\operatorname{sgn}t\,\phi'(x+t)\int_0^1\frac{ut-\sin ut}{u^2p(u)}\,du\,dt-$$

$$-\lim_{T\to\infty}\frac{1}{\pi}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)^2\phi''(x+t)\int_0^1\frac{ut-\sin ut}{u^2p(u)}\,du\,dt.$$

Now the first term on integrating by parts gives

$$\lim_{T\to\infty}\frac{2}{\pi T^2}\int_{-T}^{T}\phi(x+t)\int_0^1\frac{ut-\sin ut}{u^2p(u)}\,du\,dt-$$

$$-\lim_{T\to\infty}\frac{2}{\pi T}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\operatorname{sgn}t\,\phi(x+t)\int_0^1\frac{1-\cos ut}{up(u)}\,du\,dt.$$

Both these tend to zero, the first by (ii) and the second by (i) of the lemma proved above. Hence the R.H.S. of (6.7.10)

$$= -\lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right)^2 \phi''(x+t) \int_0^1 \frac{ut - \sin ut}{u^2 p(u)}\, du\, dt$$

$$= \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right)^2 f_h^*(x+t) \int_0^1 \frac{ut - \sin ut}{u^2 p(u)}\, du\, dt$$

by (6.7.8). Substituting this in (6.7.9) we get

$$f(x) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} f_h^*(x+t) \frac{\cos t}{t} + \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right) f_h^*(x+t) \int_0^1 \frac{ut - \sin ut}{u^2 p(u)}\, du\, dt$$

$$= -\frac{1}{\pi} \lim_{T\to\infty} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right)^2 f_h^*(x+t) \left\{ \frac{\cos t}{t} - \int_0^1 \frac{ut - \sin ut}{u^2 p(u)}\, du \right\} dt. \qquad \ldots\ldots (6.7.11)$$

Now as $$\int_0^1 \frac{ut - \sin ut}{u^2 p(u)} = \frac{1}{D^2} \int_0^1 \frac{\sin ut}{p(u)}\, du,$$

and $\int_0^1 \frac{\sin ut}{p(u)}\, du$ could be written as

$$\frac{1}{p(iD)} \int_0^1 \sin ut\, du,$$

since p(u) is even and $\neq 0$ in (0,1). We may write

$$\int_0^1 \frac{ut - \sin ut}{u^2 p(u)}\, du = -\frac{1}{(iD)^2 p(iD)} \frac{1 - \cos t}{t}$$

$$= -\frac{1}{h(iD)} \frac{1 - \cos t}{t}.$$

(6.7.11) can, therefore more appropriately, be expressed as

$$f(x) = -\frac{1}{\pi} \lim_{T\to\infty} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right)^2 f_h^*(x+t) \left\{ \frac{\cos t}{t} + \frac{1}{h(iD)} \frac{1-\cos t}{t} \right\} dt.$$

This completes the proof of the theorem for the case II.

(6.8) In the preceding chapter we discussed the inversion of the Boas transform generated by

$$h(x) = a_1 x^{2n-1} + a_2 x^{2n-3} + \ldots\ldots\ldots + a_n x,$$

where $a_n \neq 0$, and $h(x)$ not vanishing at any other point in $(0,1)$. We are now in a psition to discuss the case $a_n = 0$, i.e. when

$$h(x) = x^3 p(x),$$

$p(x)$ even and non-zero in $(0,1)$. We show here that

THEOREM. If $f(x) \in L^2$, and $h(x)$ be as defined above, then

$$f_h^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + i h(iD) \frac{\sin t}{t} \right\} dt, \qquad \ldots\ldots(6.8.1)$$

and inversely

$$f(x) = -\frac{1}{\pi} \lim_{T\to\infty} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right)^2 f_h^*(x+t) \left\{ \frac{\cos t}{t} + \frac{i}{h(iD)} \frac{\sin t}{t} \right\} dt. \qquad \ldots\ldots(6.8.2)$$

Proof. (6.8.1) is obvious from (5.5.3). For (6.8.2), following the arguments of the case II of (6.7), we find that

$$\sqrt{\frac{2}{\pi}} \int_0^1 \frac{\cos ut}{p(u)}\, du \;\varepsilon\; L^2$$

with

$$\left.\begin{array}{ll} 1/p(x) & (|x| \le 1) \\ 0 & (|x| > 1) \end{array}\right\} \qquad \ldots.(6.8.3)$$

as its Fourier transform. Again as in Chapter 5

$$F(x) = \left.\begin{array}{ll} \dfrac{i\,\mathrm{sgn}\,x\, \mathcal{F}(x)}{h(|x|)} & (|x| \le 1) \\ i\ \mathrm{sgn}\,x\, \mathcal{F}(x) & (|x| > 1) \end{array}\right\}$$

$$= \left.\begin{array}{ll} \dfrac{i\,\mathcal{F}(x)}{x^3 p(x)} & (|x| \le 1) \\ i\,\mathrm{sgn}\,x\, \mathcal{F}(x) & (|x| > 1), \end{array}\right\} \qquad \ldots.(6.8.4)$$

by the definition of h(x). Hence as in (6.7)

$$\Phi(x) = i\,\frac{\mathcal{F}(x)}{x^3} \qquad \ldots.(6.8.5)$$

and $x^3 \Phi(x)$ both belong to L^2, and so by (5.2.10)

$$f_h^*(x) = -\phi'''(x), \qquad \ldots.(6.8.6)$$

where ϕ is the inverse Fourier transform of Φ. Hence again as in (6.7)

$$\int_{-\infty}^{\infty} f(x+t)\,\frac{\sin t}{t}\,dt = \int_{-\infty}^{\infty} \phi(x+t) \int_0^1 \frac{\cos ut}{p(u)}\,du\,dt$$

$$= \lim_{T\to\infty} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right)^2 \phi(x+t) \int_0^1 \frac{\cos ut}{p(u)}\,du\,dt$$

$$= -\lim_{T\to\infty} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right)^2 \phi'''(x+t) \int_0^1 \frac{ut - \sin ut}{u^3 p(u)}\,du\,dt,$$

which is obtained by employing the patterns of the proofs of (5.4) and (6.7) and the following lemma.

LEMMA.

$$\text{(i)} \qquad \lim_{T\to\infty} \frac{1}{T}\int_1^T \phi(x\pm t)\int_0^1 \frac{1-\cos ut}{u^2 p(u)}\,du\,dt = 0.$$

$$\text{(ii)} \qquad \lim_{T\to\infty} \frac{1}{T^2}\int_1^T \phi(x\pm t)\int_0^1 \frac{ut-\sin ut}{u^3 p(u)}\,du\,dt = 0.$$

This can be proved easily by using the lemma of (5.3) and the fact that $1/p(x)$ is bounded in $(0,1)$.

Now substituting (6.8.6) we get

$$\int_{-\infty}^{\infty} f(x+t)\frac{\sin t}{t}\,dt = \lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)^2 f_h^*(x+t)\int_0^1 \frac{ut-\sin ut}{u^3 p(u)}\,du\,dt$$

$$= \lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)^2 f_h^*(x+t)\frac{i}{h(iD)}\frac{\sin t}{t}\,dt,$$

since, as in (6.7)

$$\frac{1}{p(iD)}\int_0^1 \cos ut\,du = \int_0^1 \frac{\cos ut}{p(u)}\,du,$$

$$\frac{1}{D^3}\int_0^1 \frac{\cos ut}{p(u)}\,du = \int_0^1 \frac{ut-\sin ut}{u^3 p(u)}\,du = -\frac{i}{h(iD)}\int_0^1 \cos ut\,du.$$

Substituting this, therefore, in (6.7.9), which holds in this case also, we get (6.8.2). Hence the proof of the theorem.

CHAPTER VII

SOME OTHER RESULTS

(7.1) We have seen in the two previous chapters that, if $f \in L^2$, and ν be any positive integer, f_ν^* exists for almost all x and belongs to L^2. We now prove a theorem which connects f_ν^* with $f_{\nu-1}^*$ and may be called a generalisation of Goldberg [5] Theorem 1 (Theorem (E) in the Introduction), namely

$$f_1^* = g - g.o.\delta \qquad \ldots.(7.1.1)$$

Here in (7.1.1) Goldberg has expressed f_1^* in terms of g which, as already shown (cf. p.66), can be expressed as f_o^*.

THEOREM. If f(x) and $f_\nu^*(x)$ be as defined above then for all positive integral values of ν

$$f_\nu^* = f_{\nu-1}^* - f_{\nu-1}^*.o.\delta \qquad \ldots.(7.1.2)$$

where $\delta(t)$ is the same function as in (7.1.1), namely

$$\sqrt{\frac{2}{\pi}}\ \frac{1-\cos t}{t^2}\ .$$

Proof. (First Method). By definition, we have

$$f_\nu^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 u^\nu \sin ut\, du \right\} dt .$$

Hence

$$f_\nu^*(x) = f_{\nu-1}^*(x) + \frac{1}{\pi} \int_{-\infty}^{\infty} f(x+t) \left\{ \int_0^1 u^{\nu-1}(u-1)\sin ut\, du \right\} dt .$$

$\ldots.(7.1.3)$

Now as defined

$$\delta(t) = \sqrt{\frac{2}{\pi}}\ \frac{1-\cos t}{t^2} = \sqrt{\frac{2}{\pi}} \int_0^1 (1-u)\cos ut\, du .$$

Hence
$$\delta^{(\nu)}(t) = (-1)^{\frac{\nu}{2}} \sqrt{\frac{2}{\pi}} \int_0^1 u^\nu (1-u)\cos ut\, du \quad \nu \text{ even}$$
$$= (-1)^{\frac{\nu+1}{2}} \sqrt{\frac{2}{\pi}} \int_0^1 u^\nu (1-u)\sin ut\, du \quad \nu \text{ odd}.$$

As $\Delta(t)$ (cf. (2.1.2)), the Fourier transform of $\delta(t)$, vanishes for $|t| > 1$, then

$$\delta_\nu^*(t) = -(i)^{\nu+1}\delta^{(\nu)}(t) \qquad \nu \text{ odd} \quad \text{(by (5.2.11))}$$
$$= i^\nu (H\delta)^{(\nu)}(t) \qquad \nu \text{ even} \quad \text{(by (6.1.4))}$$

where $H\delta$ stands for the Hilbert transform of δ. Thus in both the cases

$$\delta_\nu^*(t) = \sqrt{\frac{2}{\pi}} \int_0^1 u^\nu (u-1)\sin ut\, du .$$

Substituting this in (7.1.3) we get

$$f_\nu^*(x) = f_{\nu-1}^*(x) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x+t)\, \delta_{\nu-1}^*(t)\, dt$$
$$= f_{\nu-1}^*(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_\nu^*(x+t)\, \delta(t)\, dt \quad \text{(by Theorem(3.7))}$$
$$= f_{\nu-1}^*(x) - f_{\nu-1}^* \cdot 0 \cdot \delta .$$

(Second Method). As usual let $\mathcal{F}_\nu$ be the Fourier transform of

f_ν^*, then as is known

$$\mathcal{F}_\nu(x) = -i\,\mathrm{sgn}\,x\, F(x)\{1-\Delta_\nu\}$$

where $$\Delta_\nu = \begin{cases} 1-|x|^\nu & (|x|\le 1) \\ 0 & (|x|>1). \end{cases}$$

Hence $$\mathcal{F}_\nu = \mathcal{F}_{\nu-1} - i\,\mathrm{sgn}\,x\, F\{\Delta_{\nu-1}-\Delta_\nu\}$$

$$= \mathcal{F}_{\nu-1} + i\,\mathrm{sgn}\,x\, F\{1-\Delta_{\nu-1}\}\Delta,$$

since $$\Delta_{\nu-1}-\Delta_\nu = \begin{cases} |x|^\nu - |x|^{\nu-1} & (|x|\le 1) \\ 0 & (|x|>1) \end{cases}$$

$$= \begin{cases} |x|^{\nu-1}\{|x|-1\} & (|x|\le 1) \\ 0 & (|x|>1) \end{cases}$$

and so $$= -\Delta\{1-\Delta_{\nu-1}\},$$

$$\mathcal{F}_\nu = \mathcal{F}_{\nu-1} - \mathcal{F}_{\nu-1}\Delta.$$

Taking the inverse Fourier transforms we get the required result.

(7.2) As (7.1.2) is true for all positive integral values of ν, we can also write it as

$$f_\nu^* = g - \nu\, g.o.\delta + \frac{\nu(\nu-1)}{2!}\, g.o.\delta_2 - \cdots\cdots + (-1)^\nu g.o.\delta_\nu \qquad \ldots\ldots(7.2.1)$$

where δ_ν are as defined in (0.11).

Proof. Substituting the value of $f_{\nu-1}^*$ in terms of $f_{\nu-2}^*$ in (7.1.2) we get

$$f_\nu^* = f_{\nu-2}^* - 2 f_{\nu-2}^*.o.\delta + f_{\nu-2}^*.o.\delta_2 .$$

Repeating this process we obtain

$$f_\nu^* = f_0^* - \nu f_0^* \cdot 0 \cdot \delta + \cdots\cdots\cdots\cdots + (-1)^\nu f_0^* \cdot 0 \cdot \delta_\nu$$

and remembering that $f_0^* = g$ we get (7.2.1).

It can be seen now that Theorem (E) is a particular case of Theorem (7.1) and can be obtained by putting $\nu = 1$ in either (7.1.2) or (7.2.1).

(7.3) The above two formulae can also be extended to the case when the g.f. is a power of h(x), where h(x) is a continuous function of bounded variation on (0,1). The two corresponding formulae in this case will be

$$f_{(\nu)}^* = f_{(\nu-1)}^* - f_{(\nu-1)}^* \cdot 0 \cdot \rho \qquad \ldots\ldots(7.3.1)$$

and

$$f_{(\nu)}^* = g - \nu g \cdot 0 \cdot \rho + \cdots\cdots (-1)^\nu g \cdot 0 \cdot \rho_\nu \qquad \ldots\ldots(7.3.2)$$

where ρ is as defined in (3.8.1) and ρ_k are the same functions of ρ as δ_k are of δ, and

$$f_{(\nu)}^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 \{h(u)\}^\nu \sin ut \, du \right\} dt.$$

Proof. (7.3.1) is already proved for $\nu = 1$, as Theorem (3.8). For the general case we follow the pattern of the second proof of (7.1) and get as in Chapters 5 and 6

$$\mathcal{F}_{(\nu)}(x) = -i\,\mathrm{sgn}\,x\,F(x)\left\{1-\Delta_{(\nu)}\right\}$$

where $\mathcal{F}_{(\nu)}$ is the Fourier transform of $f^*_{(\nu)}$, and

$$\Delta_{(\nu)} = \begin{cases} 1-\{h(|x|)\}^{\nu} & (|x|\leq 1) \\ 0 & (|x|>1) \end{cases}$$

Thus as in (7.1)

$$\mathcal{F}_{(\nu)} = \mathcal{F}_{(\nu-1)} - \mathcal{F}_{(\nu-1)}\,\Delta_{(1)}$$

where $\Delta_{(1)}$ is Δ_h of (3.4.5) and, as in (3.8), is the Fourier transform of ρ. Hence on taking the inverse Fourier transforms we get (7.3.1). For (7.3.2) we substitute the value of $f^*_{(\nu-1)}$ in terms of $f^*_{(\nu-2)}$ and then the value of $f^*_{(\nu-2)}$ in terms of $f^*_{(\nu-3)}$ and so on, and here also remembering that

$$f^*_{(0)} = g$$

we get (7.3.2).

(7.4) Lastly we discuss the transform

$$q(x) = \frac{1}{\pi} P\int_{-\infty}^{\infty} f(x+t)\,\frac{e^{it}}{t}\,dt, \qquad \ldots.(7.4.1)$$

which is immediately obtained from the Hilbert transform formula, since we notice that

$$e^{ix}q(x)$$

is the Hilbert transform of $e^{ix}f(x)$. We may also write (7.4.1) as

$$g(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t)\left\{\frac{\cos t}{t} + i\int_0^1 \cos ut\, du\right\} dt$$

which is a particular case of

$$f^{\Phi}(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t)\left\{\frac{\cos t}{t} + i\int_0^1 h(u)\cos ut\, du\right\} dt \quad \ldots\ldots(7.4.2)$$

where, as before, h(x) is a continuous function of bounded variation on (0,1). If now $\mathcal{F}^{\Phi}$ be the Fourier transform ($\mathcal{F}^{\Phi}$ is <u>not</u> the (7.4.2) transform of $\mathcal{F}$) of f^{Φ}, then

$$\mathcal{F}^{\Phi}(x) = \begin{cases} i\,h(|x|)\,F(x) & (|x|\le 1) \\ -i\,\mathrm{sgn}\,x\,F(x) & (|x|>1), \end{cases} \quad \ldots\ldots(7.4.3)$$

since $\sqrt{\frac{2}{\pi}}\int_0^1 h(u)\cos ut\, du \in L^2$ with

$$\begin{cases} h(|x|) & (|x|\le 1) \\ 0 & (|x|>1) \end{cases}$$

as its Fourier transform.

Thus here also, if F(x) vanishes a.e. on $|x| \le 1$,

$$\mathcal{F}^{\Phi}(x) = -i\,\mathrm{sgn}\,x\,F(x)$$

or

$$f^{\Phi} = g \quad \ldots\ldots(7.4.4)$$

and thus we see that Theorem (D) is true for transform (7.4.2). We may also express (7.4.4) as

$$f^{\Phi\Phi} = -f$$

a result analogous to (0.8).

If f and $\phi \in L^2$, and f^{Φ} and ϕ^{Φ} be their (7.4.2) transforms then it can be verified easily that

$$\int_{-\infty}^{\infty} f(x)\,\phi^{\Phi}(x)\,dx = \int_{-\infty}^{\infty} f^{\Phi}(x)\,\phi(x)\,dx .$$

~~If $f = \phi$, then~~

$$\int_{-\infty}^{\infty} f(x)\,f^{\Phi}(x)\,dx = \theta ,$$

~~and if $\phi = f^{\Phi}$, then~~

$$\int_{-\infty}^{\infty} \{f^{\Phi}(x)\}^2\,dx = \int_{-\infty}^{\infty} f(x)\,f^{\Phi\Phi}(x)\,dx$$

a result similar to Heywood [6] Theorem 3 (Theorem (J) of the Introduction).

(7.5) We now study the special case of the transform (7.4.2) when $h(x) = x^{\nu}$, where ν is any positive integer, and denote the transform with x^{ν} as the g.f. by f_{ν}^{Φ}. It can be seen now that the transform (7.4.1) is in fact f_0^{Φ}. We prove, as in (7.1), that

THEOREM.

$$f_{\nu}^{\Phi} = f_{\nu-1}^{\Phi} - f_{\nu-1}^{\Phi} \cdot o \cdot \delta \qquad \ldots.(7.5.1)$$

and

$$f_{\nu}^{\Phi} = q - \nu q \cdot o \cdot \delta + \ldots\ldots + (-1)^{\nu} q \cdot o \cdot \delta_{\nu} \qquad \ldots.(7.5.2)$$

Proof. If $\mathcal{F}_{\nu}^{\Phi}$ be the Fourier transform of f_{ν}^{Φ} then by (7.4.3)

$$\mathcal{F}_\nu^{\Phi}(x) = \begin{cases} i\,|x|^\nu F(x) & (|x|\le 1) \\ -i\,\mathrm{sgn}\,x\,F(x) & (|x|>1) \end{cases} \quad \ldots\ldots(7.5.3)$$

and so $$\mathcal{F}_\nu^{\Phi} = \mathcal{F}_{\nu-1}^{\Phi} - i\,|x|^{\nu-1} F(x)\,\Delta(x)$$

where $\Delta(x)$ is as given in (2.1.2). Thus

$$\mathcal{F}_\nu^{\Phi} = \mathcal{F}_{\nu-1}^{\Phi} - \mathcal{F}_{\nu-1}^{\Phi}\,\Delta .$$

Taking the inverse Fourier transforms of both the sides we get (7.5.1). (7.5.2) can now be obtained in the similar way as (7.2.1).

Both these formulae may also be extended, as in (7.3), so as to cover the case when the g.f. is a power of h(x), we have then

$$f_{(\nu)}^{\Phi} = f_{(\nu-1)}^{\Phi} - f_{(\nu-1)}^{\Phi} \cdot o \cdot \rho$$

and

$$f_{(\nu)}^{\Phi} = q - \nu\, q \cdot o \cdot \rho + \frac{\nu(\nu-1)}{2!} q \cdot o \cdot \rho_2 - \cdots + (-1)^\nu q \cdot o \cdot \rho_\nu$$

where $\rho(x)$ is as in (7.3) or (3.8), and

$$f_{(\nu)}^{\Phi}(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + \int_0^1 \{h(u)\}^\nu \cos ut\, du \right\} dt .$$

Results similar to those of Chapters 5 and 6 can also be derived, viz

$$f_\nu^{\Phi}(x) = -(iD)^\nu \psi_1(x) + \psi_2(x) \qquad \nu \text{ odd},$$

$$= i(iD)^{\nu}\phi_1(x) + \psi_2(x) \qquad \nu \text{ even}$$

or $$f_{\nu}^{\phi}(x) = -(iD)^{\nu}\psi(x) \qquad \nu \text{ odd},$$

$$= i(iD)^{\nu}\phi(x) \qquad \nu \text{ even}.$$

In general

$$f_h^{\phi}(x) = -h(iD)\psi_1 + \psi_2 \qquad h(x) \text{ odd function},$$

$$= ih(iD)\phi_1 + \psi_2 \qquad h(x) \text{ even function},$$

where ψ is the Hilbert transform of ϕ, and similarly for ψ_1.

Further

$$\Phi(x) = F(x) \qquad (|x| \le 1),$$

$$\Phi_1(x) = \begin{cases} F(x) & (|x| \le 1) \\ 0 & (|x| > 1) \end{cases}$$

and

$$\Psi_2(x) = \begin{cases} 0 & (|x| \le 1) \\ G(x) & (|x| > 1), \end{cases}$$

and $h(1) = 1$.

If $F(x)$ vanishes a.e. for $|x| > 1$, then

$$f_h^{\phi}(x) = -h(iD)\,g(x) \qquad h \text{ odd},$$

$$= ih(iD)f(x) \qquad h \text{ even}.$$

<u>Illustrations</u>. $h(x) = x$, and $F(x) = 0$ a.e $(|x| > 1)$.

(i) $\nu = 0$, $f_0^{\phi}(x) = g(x) = if(x)$.

(ii) $\nu = 1$, $f_1^{\Phi}(x) = -ig'(x)$.

(iii) $\nu = 2$, $f_2^{\Phi}(x) = -if''(x)$.

Here we notice that for odd values of ν we get a relation between the transform (7.4.2) and the derivatives of $g(x)$, and in case of even values the relation is between f_ν^{Φ} and $f^{(\nu)}$. If we recall the corresponding results in connection with the Boas transforms, as given in Chapters 5 and 6, we find that the situation is just the reverse there. Thus we may say that the transform (7.4.2) fills the gap which we find in the study of the Boas transform. This aspect is even more evident when we try to invert the transform (7.4.2). Here the formulae for inversion for the different cases would be as follows :-

(i) $\nu = 0$,

$$-f(x) = \frac{1}{\pi} P\int_{-\infty}^{\infty} g(x+t)\left\{\frac{\cos t}{t} + i\int_0^{\infty} \cos ut\, du\right\} dt$$

or $$-f(x) = \frac{1}{\pi} P\int_{-\infty}^{\infty} g(x+t)\frac{e^{it}}{t}\, dt.$$

(ii) $\nu = 1$ or 3,

$$-f(x) = \frac{1}{\pi}\lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)^{\nu} f_\nu^{\Phi}(x+t)\left\{\frac{\cos t}{t} + (iD)^{-\nu}\frac{1-\cos t}{t}\right\} dt.$$

(iii) $\nu = 2$,

$$-f(x) = \frac{1}{\pi}\lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)^{\nu-1} f_\nu^{\Phi}(x+t)\left\{\frac{\cos t}{t} + i(iD)^{-\nu}\frac{\sin t}{t}\right\} dt.$$

It will not be out of place, here, to write the two direct formulae expressing f_ν^ϕ in terms of f(x) and the differential operator D, namely

$$f_\nu^\phi(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + (iD)^\nu \frac{1-\cos t}{t} \right\} dt \qquad \nu \text{ odd},$$

and $$f_\nu^\phi(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + i(iD)^\nu \frac{\sin t}{t} \right\} dt \quad \nu \text{ even}.$$

Comparing these with (6.3.1) and (5.3.2) we find that the latter defines f_ν^* if ν is odd and f_ν^ϕ if ν is even, and the former defines f_ν^ϕ if ν is odd otherwise f_ν^*.

(7.6) In general we may say that

$$f^*(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(x+t) \left\{ \frac{\cos t}{t} + \phi(t) \right\} dt$$

exists a.e. and $\in L^2$, if f(x) and $\phi(x) \in L^2$, and are such that

$$F(x)\,\overline{\Phi}(x) \text{ also } \in L^2.$$

Now if $F(x) = 0$ a.e. $(|x| \leq 1)$ and $\overline{\Phi} = 0$ a.e $(|x| > 1)$

then $$f^*(x) = g(x).$$

In case of the Hilbert transform

$$\phi(t) = \frac{1-\cos t}{t}$$

and for the Goldberg transform

$$\phi(t) = \frac{\sin t}{t^2} - \frac{\cos t}{t}$$

The condition that $\phi(t) \in L^2$ is by no means necessary for f^* to belong to L^2. What we need here is that

$$P \int_{-\infty}^{\infty} f(x+t)\phi(t)\,dt \in L^2.$$

CHAPTER VIII

ALMOST PERIODIC FUNCTIONS AND THE GOLDBERG TRANSFORM

(8.1) As in Cossar [7], let T be the set of trigonometric polynomials (t.p)

$$\sum_{r=1}^{n} k_r e^{i\lambda_r x}$$

where k_r are complex and λ_r real constants, and U be the set of functions each of which is the limit of a uniformly convergent sequence of the functions of T. The functions of U are called uniformly almost periodic or Bohr almost periodic functions. It could be seen that the set U possesses the following properties (not a complete account of course).

(i) $U \supset T$, in fact T is a proper subset of U.

(ii) If $f \in U$, then for each $\epsilon > 0$ $\exists$ a t.p $t(x)$ such that

$$|f(x) - t(x)| < \epsilon \qquad -\infty < x < \infty ,$$

and conversely.

(iii) If $f \in U$, then f is bounded.

(iv) If $f \in U$, then f is uniformly continuous in $-\infty < x < \infty$.

(v) For any function $f \in U$, the mean value

$$M(f) = \lim_{\omega \to \infty} \frac{1}{\omega} \int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} f(x)\,dx$$

exists. Also

$$M\{ f(x) e^{-i\lambda x} \} \qquad (\lambda \text{ real})$$

is zero except when λ belongs to a certain denumerable set. These exceptional values $\lambda_1, \lambda_2, \ldots\ldots$ are called the exponents of f, and the formal expression

$$\sum_r c_r e^{i\lambda_r x}$$

is called the Fourier series of f, where

$$c_r = M\{f(x)e^{-i\lambda_r x}\}$$

(vi) If $f_n \in U$ $(n = 1,2,\ldots\ldots)$, and f_n tends uniformly to f for all x then $f \in U$, and

$$M\{f_n e^{-i\lambda x}\} \longrightarrow M\{f(x)e^{-i\lambda x}\}$$

as $n \longrightarrow \infty$.

The series

$$i\sum_{r=1}^{\infty} c_r \operatorname{sgn}\lambda_r e^{i\lambda_r x} \qquad \ldots.(8.1.1)$$

is called the conjugate series of the Fourier series of f(x). Our investigations, here in this chapter, are mainly concerned with the behaviour of the conjugate series.

We put down here some of the corresponding results from Cossar's lecture notes [7] on the summability and convergence of the Fourier series of f(x), when $f \in U$.

THEOREM (N). If f is even and belongs to U, and its exponents $0, \lambda_1, \lambda_2, \ldots$ are such that each finite interval contains only a finite

number of them, then the Fourier series of f(x) is summable (R, λ , 1) to f(x). The summability here is uniform. We may also write the above statement as follows.

If $$\Sigma_\lambda(x) = c_0 + 2\sum_{\lambda_\nu \le \lambda} \left(1 - \frac{\lambda_\nu}{\lambda}\right) c_\nu \cos \lambda_\nu x \quad (\lambda_\nu > 0)$$

then $\sum_\lambda(x) \in U$, and tends uniformly to f(x) as $\lambda \to \infty$,

....(8.1.2)

and $$\Sigma_\lambda(x) = \frac{1}{\lambda\pi} \int_{-\infty}^{\infty} f(x+u) \frac{1-\cos \lambda u}{u^2} du .$$

DEFINITION. We say that the series

$$\sum_1^\infty c_\nu \phi(\lambda_\nu)$$

is summable (R, λ , 1) i.e. summable in the sense of M. Riesz, if

$$\sum_{0<\lambda_\nu<\lambda} \left(1 - \frac{\lambda_\nu}{\lambda}\right) c_\nu \phi(\lambda_\nu)$$

tends to a limit as $\lambda \to \infty$. If $\lambda_n = n$, (R, λ , 1) $\equiv$ (C,1).

THEOREM (O). *If* (i) f $\in$ U, f is even with exponents

$$0, \pm\lambda_1, \pm\lambda_2, \ldots\ldots\ldots$$

where $$0 < \lambda_1 < \lambda_2 < \ldots\ldots \quad \lambda_n \longrightarrow \infty .$$

(ii) f satisfies D for x = 0.

(iii) The Fourier series converges to f(0) at x = 0.

Then the Fourier series converges to f(x) at any point at which D is satisfied.

Condition D. f(x) satisfies D at x if for some $\delta > 0$

$$\frac{1}{\pi}\int_0^{\delta}\{f(x+u)+f(x-u)\}\frac{\sin \lambda u}{u}\,du$$

tends to f(x) as $\lambda \longrightarrow \infty$.

(8.2) THEOREM. If $f \in U$, and is such that each finite interval contains only a finite number of its exponents, the open interval (-1,1) containing none except possibly 0, then the conjugate series (8.1.1) is summable (R, λ , 1) to $f^*(x)$ for almost all x, where

$$f^*(x) = \frac{1}{\pi} P\int_{-\infty}^{\infty} f(x+u)\frac{\sin u}{u^2}\,du \qquad \ldots.(8.2.1)$$

is the Goldberg transform of f(x).

Proof. We first show that (8.2.1) exists a.e.

Let $\delta > 0$, and

$$f_\delta(u) = f(u) \qquad |u| \le \delta$$
$$= 0 \qquad \text{elsewhere.}$$

When $|x| < \delta$ the integrals

$$\int_{-\infty}^{-\delta} f(u)\frac{\sin(u-x)}{(u-x)^2}\,du\ , \quad \int_{\delta}^{\infty} f(u)\frac{\sin(u-x)}{(u-x)^2}\,du$$

are absolutely convergent, and

$$\frac{1}{\pi}\int_{-\delta}^{\delta} f(u)\,\frac{\sin(u-x)}{(u-x)^2}\,du = \frac{1}{\pi}\int_{-\infty}^{\infty} f_\delta(u)\,\frac{\sin(u-x)}{(u-x)^2}\,du$$

exists for almost all x by Goldberg [5] Theorem 1 (Theorem (E) in the Introduction), since $f_\delta(x) \in L^2$. Hence we get the existence of (8.2.1) for almost all x.

Let the exponents of f(x) be

$$0 = \lambda_0, \lambda_1, \lambda_2, \ldots\ldots\ldots\ldots$$

such that $\quad 1 \leq |\lambda_1| < |\lambda_2| < |\lambda_3| < \ldots\ldots\ldots$

We now prove two lemmas.

LEMMA 1. If

$$\bar{\Sigma}_\lambda(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x+u)\,\varphi_\lambda(u)\,du \qquad \ldots\ldots(8.2.2)$$

where $\quad \varphi_\lambda(u) = \frac{\sin u}{u^2} - \frac{\sin u}{\lambda u^2} - \frac{\sin \lambda u}{\lambda u^2} + \frac{2(1-\cos u)}{\lambda u^3}$,

and f(x) is as defined above, then $\bar{\Sigma}_\lambda(x)$ also belongs to U, and is equal to

$$i\sum_{|\lambda_r| \leq \lambda} \operatorname{sgn}\lambda_r\, c_r \left(1 - \frac{|\lambda_r|}{\lambda}\right) e^{i\lambda_r x}.$$

The integral on the right of (8.2.2) is absolutely convergent, since $\varphi_\lambda(u) \in L$, and f(x) is bounded. Hence $\bar{\Sigma}_\lambda(x)$ exists for all x.

Let $\quad t(x) = \sum_{r=1}^{n} k_r e^{i\rho_r x}$

with $|\rho_r| \geq 1$ for all r except possibly $\rho_r = 0$ for some r, be

a t.p such that for all x

$$\left| f(x) - t(x) \right| < \epsilon .$$

The restriction on ρ_r's could be justified by the fact that the Fourier series of f(x) is defined as the formal term by term limit of the sequence $\{t(x)\}$ and $|\lambda_r| \geq 1$ (r = 1,2,.....). If we write

$$\bar{T}_m(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} t(x+u)\, \varphi_\lambda(u)\, du ,$$

then the absolute convergence of (8.2.2) gives

$$\left| \bar{\Sigma}_\lambda(x) - \bar{T}_m(x) \right| < K\epsilon \qquad \text{(all x)}, \quad K = K(\lambda).$$

To complete the proof of the lemma we need prove only that $\bar{T}_m(x)$ is a t.p. This is obvious from the fact that

$$\bar{T}_m(x) = \frac{1}{\pi} \sum_1^n k_r e^{i\rho_r x} \int_{-\infty}^{\infty} e^{i\rho_r u} \varphi_\lambda(u)\, du .$$

Now the R.H.S. can be written as

$$\sum_1^n k_r e^{i\rho_r x} \left[\frac{1}{\pi}\left(1 - \frac{1}{\lambda}\right) \int_{-\infty}^{\infty} \frac{e^{i\rho_r u} \sin u}{u^2}\, du \right.$$

$$\left. - \frac{1}{\lambda\pi} \int_{-\infty}^{\infty} \frac{e^{i\rho_r u} \sin \lambda u}{u^2}\, du + \frac{2}{\lambda\pi} \int_{-\infty}^{\infty} e^{i\rho_r u} \frac{1-\cos u}{u^3}\, du \right],$$

since the separate integrals exist, as can be seen as follows.

Calling the three terms within the bracket by I_1, I_2 and I_3, we find that

$$I_1 = -i\left(\frac{1}{\lambda} - 1\right) \operatorname{sgn} \rho_r \qquad \text{(since } |\rho_r| \geq 1, \text{ and } |\rho_r| < \lambda).$$

$$I_2 = -\frac{i}{\lambda}\rho_r .$$

$$I_3 = \frac{2i}{\lambda\pi}\int_{-\infty}^{\infty}\frac{\sin\rho_r u}{u}\cdot\frac{1-\cos u}{u^2}\,du,$$

since $$\int_{-\infty}^{\infty}\frac{\cos\rho_r u\,(1-\cos u)}{u^3}\,du = 0.$$

We may write

$$I_3 = \frac{2i}{\lambda\pi}\int_{-\infty}^{\infty}\frac{\sin\rho_r u}{u}\left[\frac{1}{2}\int_{-1}^{1}\{1-|x|\}e^{-ixu}\,dx\right]du,$$

since the Fourier transform of

$$\frac{1-\cos x}{x^2}$$

is $$\left.\begin{array}{ll}\sqrt{\frac{\pi}{2}}\{1-|x|\} & (|x|\le 1)\\ 0 & (|x|>1).\end{array}\right\}$$

Hence $$I_3 = \frac{i}{\lambda\pi}\int_{-1}^{1}\{1-|x|\}dx\int_{-\infty}^{\infty}\frac{\sin\rho_r u}{u}e^{-ixu}\,du \quad \ldots.(8.2.3)$$

$$= \frac{i}{\lambda\pi}\int_{-1}^{1}\{1-|x|\}\pi\,\mathrm{sgn}\,\rho_r\,dx = \frac{i}{\lambda}\,\mathrm{sgn}\,\rho_r .$$

Inversion justified on the grounds that the R.H.S. of (8.2.3)

$$= \frac{i}{\lambda\pi}\lim_{a\to\infty}\int_{-1}^{1}\{1-|x|\}\int_{-a}^{a}\frac{\sin\rho_r u}{u}e^{-ixu}\,du\,dx$$

$$= \frac{i}{\lambda\pi}\lim_{a\to\infty}\int_{-a}^{a}\frac{\sin\rho_r u}{u}\,du\int_{-1}^{1}\{1-|x|\}e^{-ixu}\,dx .$$

Thus $$I_1 + I_2 + I_3 = i\,\mathrm{sgn}\,\rho_r\left(1-\frac{|\rho_r|}{\lambda}\right)$$

or $$\bar{\tau}_n(x) = i\sum_{1}^{n}\mathrm{sgn}\,\rho_r\left(1-\frac{|\rho_r|}{\lambda}\right)k_r e^{i\rho_r x}.$$

Hence we find that $\bar{\Sigma}_\lambda(x)\varepsilon U$, as it is the uniform limit of the

sequence of t.ps $\bar{\tau}_n(x)$, and hence the Fourier series of $\bar{\Sigma}_\lambda(x)$ is the formal limit of $\bar{\tau}_n(x)$. But as there are only a finite number of the exponents in $(-\lambda, \lambda)$, $\bar{\Sigma}_\lambda(x)$ is itself a t.p, and is therefore equal to

$$i \sum_{|\lambda_r| \le \lambda} \operatorname{sgn} \lambda_r \left(1 - \frac{|\lambda_r|}{\lambda}\right) c_r e^{i\lambda_r x}.$$

This completes the proof of Lemma 1.

LEMMA 2. If $f \in U$, then for almost all x

$$\int_{-\infty}^{\infty} f(x+u) \frac{\sin \lambda u}{\lambda u^2} du \to 0 \quad \text{as} \quad \lambda \longrightarrow \infty.$$

Here we need prove only that for some $\delta > 0$

$$\lim_{\lambda \to \infty} \int_{-\delta}^{\delta} f(x+u) \frac{\sin \lambda u}{\lambda u^2} du \to 0 \quad \text{as} \quad \lambda \longrightarrow \infty.$$

Now

$$\int_{-\delta}^{\delta} f(x+u) \frac{\sin \lambda u}{\lambda u^2} du = \left[\int_0^{\frac{1}{\lambda}} + \int_{\frac{1}{\lambda}}^{\delta} \right] \{ f(x+u) - f(x-u) \} \frac{\sin \lambda u}{\lambda u^2} du$$

and

$$\left| \int_{\frac{1}{\lambda}}^{\delta} \{ f(x+u) - f(x-u) \} \frac{\sin \lambda u}{\lambda u^2} du \right|$$

$$\le \frac{1}{\lambda} \int_{\frac{1}{\lambda}}^{\delta} | f(x+u) - f(x-u) | \frac{du}{u^2}$$

$$\le \frac{1}{\lambda} \left[\Psi(u) u^{-2} \right]_{\frac{1}{\lambda}}^{\delta} + \frac{2}{\lambda} \int_{\frac{1}{\lambda}}^{\delta} \Psi(u) u^{-3} du$$

$$\leq \frac{\Psi(\delta)}{\lambda \delta^2} + \lambda \Psi\left(\frac{1}{\lambda}\right) + \frac{1}{\lambda} \int_{\frac{1}{\lambda}}^{\delta} o\left(\frac{1}{u^2}\right) du$$

$$\leq o(1) \quad \text{as} \quad \lambda \longrightarrow \infty$$

where $\Psi(h) = \int_0^h |f(x+u) - f(x-u)| \, du$

and by Lebesgue Theorem ([9]. (11), - p.65)

$$\Psi(h) = o(h) \quad \text{as } h \longrightarrow +0 \quad \text{for almost all x.}$$

Now

$$\int_0^{\frac{1}{\lambda}} \{f(x+u) - f(x-u)\} \frac{\sin \lambda u}{\lambda u^2} du$$

$$= \int_0^{\frac{1}{\lambda}} \{f(x+u) - f(x-u)\} \left(\frac{\sin \lambda u}{\lambda u^2} - \frac{1}{u}\right) du$$

$$+ \int_0^{\frac{1}{\lambda}} \{f(x+u) - f(x-u)\} \frac{du}{u},$$

and as $\left|\frac{\sin \lambda u}{\lambda u^2} - \frac{1}{u}\right| \leq \lambda \quad \text{for } 0 \leq u \leq \frac{1}{\lambda}$

$$\left| \int_0^{\frac{1}{\lambda}} \{f(x+u) - f(x-u)\} \left(\frac{\sin \lambda u}{\lambda u^2} - \frac{1}{u}\right) du \right|$$

$$\leq \lambda \int_0^{\frac{1}{\lambda}} |f(x+u) - f(x-u)| \, du$$

$$\leq \lambda \Psi\left(\frac{1}{\lambda}\right) \to 0 \text{ as } \lambda \longrightarrow \infty.$$

Again $\int_0^{\frac{1}{\lambda}} \{f(x+u) - f(x-u)\} \frac{du}{u}$

exists for almost all x, and so $\longrightarrow 0$ as $\lambda \longrightarrow \infty$.

This proves Lemma 2.

If now the series (8.1.1) is summable (R, λ, 1), its 'sum' is

$$\lim_{\lambda \to \infty} \bar{\Sigma}_\lambda(x)$$

or $$\lim_{\lambda \to \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(x+u) \left\{ \frac{\sin u}{u^2} - \frac{\sin u}{\lambda u^2} - \frac{\sin \lambda u}{\lambda u^2} + \frac{2}{\lambda u^3}(1-\cos u) \right\} du .$$

Considering the terms in the bracket separately we find that the first integral, being independent of λ , is by (8.2.1) $f^*(x)$. The second, of course, tends to zero as $\lambda \to \infty$,(since apart the factor $\frac{1}{\lambda}$ its integrand is same as in the first). The third tends to zero by Lemma 2. In the fourth the integrals

$$\frac{1}{\lambda} \int_{-\infty}^{-\delta} f(x+u) \frac{1-\cos u}{u^3} du , \quad \frac{1}{\lambda} \int_{\delta}^{\infty} f(x+u) \frac{1-\cos u}{u^3} du$$

being absolutely convergent for $\delta > 0$, tend to 0 as $\lambda \longrightarrow \infty$. Again

$$f(x+u) \frac{1-\cos u}{u^2} \quad \varepsilon \quad L(-\delta, \delta),$$

since $\frac{1-\cos u}{u^2} \varepsilon$ L, and f(x) is bounded, hence

$$\int_{-\delta}^{\delta} f(x+u) \frac{1-\cos u}{u^2} \cdot \frac{du}{u}$$

exists for almost all x. Thus the fourth integral also tends to zero as $\lambda \longrightarrow \infty$. Hence we find that

$$\lim_{\lambda \to \infty} \bar{\Sigma}_\lambda(x) = f^*(x)$$

for almost all x.

Corollary. If the interval (-1,1) also contains a finite number of the exponents of f(x), the conjugate series is still summable

$(R, \lambda, 1)$ (this is obvious, since the introduction of a finite number of exponents adds only a t.p to the series).

If the new exponents be

$$\mu_1, \mu_2, \ldots\ldots \mu_j$$

then $$f(x) = h(x) + \phi(x),$$

where $$h(x) = \sum_{r=1}^{j} k_r e^{i\mu_r x} \quad \text{(a t.p)},$$

and $\phi(x) \in U$ with $\lambda_1, \lambda_2, \ldots\ldots\ldots$ as its exponents. The conjugate series in this case has the 'sum' (in the M. Riesz sense)

$$k(x) + \phi^*(x),$$

where $k(x)$ is the Hilbert transform of $h(x)$, and $\phi^*(x)$ is the Goldberg transform of $\phi(x)$.

(8.3) THEOREM. If $f \in U$, is even, and the number of exponents are finite in each finite interval then the conjugate series of the Fourier series of $f(x)$ is summable $(R, \lambda, 1)$ to $g(x)$, the Hilbert transform of $f(x)$, for almost all x.

Proof. The function is even and hence satisfies Cossar [3] Theorem 1 on the existence of $g(x)$ [The existence of g does not require that $m(f)=0$].

Let the exponents of $f(x)$ be

$$0 = \mu_0, \pm\mu_1, \pm\mu_2, \ldots \pm\mu_j, \pm\lambda_1, \pm\lambda_2, \ldots\ldots\ldots$$

such that $\mu_r < 1$ for $r = 1,2,\ldots\ldots j$, and $1 \leq \lambda_r < \lambda_{r+1}$ for all r.

Let $$h(x) = k_0 + 2\sum_{r=1}^{j} k_r \cos\mu_r x$$

where k_r is the Fourier coefficient of f corresponding to the exponent μ_ν.

Now f(x) can be expressed as

$$h(x) + \phi(x),$$

where $\phi(x) \in U$, is even, and has

$$\pm \lambda_1, \pm \lambda_2, \ldots\ldots\ldots\ldots$$

as its exponents. The conjugate series (8.1.1) of $\phi(x)$ will now be

$$-2 \sum_{\nu} c_\nu \sin \lambda_\nu x$$

and by Theorem (8.2) it is summable (R, λ, 1) to $\phi^*(x)$ for almost all x, i.e.

$$\bar{\Sigma}_\lambda(x) = -2 \sum_{\lambda_\nu \leq \lambda} \left(1 - \frac{\lambda_\nu}{\lambda}\right) c_\nu \sin \lambda_\nu x \longrightarrow \phi^*(x) \text{ as } \lambda \longrightarrow \infty$$

and as h(x) is only a t.p, we find that the conjugate series of f(x) is summable (R, λ, 1) to

$$k(x) + \phi^*(x)$$

for almost all values of x, where k(x) is the Hilbert transform of h(x), and $\phi^*(x)$ is the Goldberg transform of $\phi(x)$.

In order to prove that

$$\phi^*(x) = \psi(x)$$

$\psi(x)$ being the Hilbert transform of $\phi(x)$, let us write, as in

Theorem (N),

$$\Sigma_\lambda(x) = 2 \sum_{\lambda_r \le \lambda} \left(1 - \frac{\lambda_r}{\lambda}\right) c_r \cos \lambda_r x$$

then $$\bar{\Sigma}_\lambda(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\Sigma_\lambda(u)}{u-x}\, du$$

and also $$= \frac{1}{\pi} P \int_{-\infty}^{\infty} \Sigma_\lambda(u)\, \frac{\sin(u-x)}{(u-x)^2}\, du,$$

since it could be easily verified that the Hilbert and the Goldberg transforms of a t.p are equal provided the exponents of the t.p are numerically greater than or equal to unity. Hence

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \Sigma_\lambda(u) \left\{ \frac{1}{u-x} - \frac{\sin(u-x)}{(u-x)^2} \right\} du = 0 \quad \ldots\ldots (8.3.1)$$

for all λ . Now if $|x| < c$ $(c>0)$, then

$$\lim_{\lambda \to \infty} \left(\int_{-\infty}^{-c} + \int_{c}^{\infty} \right) \Sigma_\lambda(u)\, \frac{\sin(u-x)}{(u-x)^2}\, du = \left(\int_{-\infty}^{-c} + \int_{c}^{\infty} \right) \phi(u)\, \frac{\sin(u-x)}{(u-x)^2}\, du,$$

since by (8.1.2) $\Sigma_\lambda(u)$ tends uniformly to $\phi(u)$ as $\lambda \to \infty$.

Similarly

$$\lim_{\lambda \to \infty} \int_{-c}^{c} \Sigma_\lambda(u) \left\{ \frac{1}{u-x} - \frac{\sin(u-x)}{(u-x)^2} \right\} du = \int_{-c}^{c} \phi(u) \left\{ \frac{1}{u-x} - \frac{\sin(u-x)}{(u-x)^2} \right\} du,$$

since $\left\{ \frac{1}{u-x} - \frac{\sin(u-x)}{(u-x)^2} \right\}$ is bounded in $(-c, c)$.

Lastly

$$\left(\int_{-\infty}^{-c} + \int_{c}^{\infty}\right) \Sigma_\lambda(u) \frac{du}{u-x} = 2x \int_c^\infty \frac{\Sigma_\lambda(u)}{u^2-x^2} du,$$

since $\Sigma_\lambda(u)$ is even, and as

$$\int_c^\infty \frac{du}{u^2-x^2}$$

is absolutely convergent, the expression on the right tends, as $\lambda \to \infty$, to

$$2x \int_c^\infty \frac{\phi(u)}{u^2-x^2} du = \left(\int_{-\infty}^{-c} + \int_c^\infty\right) \frac{\phi(u)}{u-x} du.$$

Combining all the above results, we get

$$\lim_{\lambda\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \Sigma_\lambda(u) \left\{ \frac{1}{u-x} - \frac{\sin(u-x)}{(u-x)^2} \right\} du$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(u) \left\{ \frac{1}{u-x} - \frac{\sin(u-x)}{(u-x)^2} \right\} du$$

$$= f(x) - \phi^*(x).$$

But by (8.3.1) the L.H.S. is zero, hence

$$f(x) = \phi^*(x).$$

Thus we get the required result.

(8.4) THEOREM.(On the convergence of (8.1.1)). If $f \in U$, is even, and the exponents satisfy the conditions of Theorem (8.2) then a necessary and sufficient condition for the convergence

of the conjugate series

$$-2\sum_{r} c_r \sin \lambda_r x \qquad (\lambda_r \geq 1)$$

is that

$$\lim_{n\to\infty} \int_0^{\delta} \{f(x+u) - f(x-u)\} \frac{\cos \lambda_n u}{u} du \quad \ldots\ldots (8.4.1)$$

exists, provided $\lambda_{n+1} - \lambda_n > h > 0$ all n.

Proof.

The existence of (8.4.1) implies the existence of

$$\int_0^{\delta} \{f(x+u) - f(x-u)\} \frac{du}{u}$$

and therefore of $f^*(x)$.

Now by Lemma 1 for Theorem (8.2)

$$2\sum_{1}^{n} \left(1 - \frac{\lambda_r}{\lambda}\right) c_r \sin \lambda_r x \qquad (\lambda_n \leq \lambda \leq \lambda_{n+1})$$

$$= \frac{1}{\lambda\pi} \int_{-\infty}^{\infty} f(x+u) \frac{-\lambda u \sin u + u \sin u + u \sin \lambda u - 2(1-\cos u)}{u^3} du.$$

Multiplying by λ, substituting $\lambda_n + h$ and λ_n for λ, and subtracting we get

$$2h \sum c_r \sin \lambda_r x = -\frac{h}{\pi} \int_{-\infty}^{\infty} f(x+u) \frac{\sin u}{u^2} du -$$

$$- \frac{1}{\pi} \int_{-\infty}^{\infty} f(x+u) \sin \lambda_n u \, \frac{1-\cos hu}{u^2} du$$

$$+ \frac{1}{\pi} \int_{-\infty}^{\infty} f(x+u) \frac{\sin hu}{u^2} \cos \lambda_n u \, du.$$

The first integral on the right is $-hf^*(x)$, the second tends to zero as $\lambda_n \to \infty$, since

$$\int_{-\infty}^{\infty} f(x+u)\,\frac{1-\cos hu}{u^2}\,du$$

is absolutely convergent. Likewise in the third integral

$$\left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty}\right) f(x+u)\,\frac{\sin hu}{u^2}\cos\lambda_n u\,du \longrightarrow 0$$

as $\lambda_n \to \infty$. Finally we get

$$\lim_{\lambda_n\to\infty} \int_0^{\delta} \left\{ f(x+u) - f(x-u) \right\} \frac{\sin hu}{u}\,\frac{\cos\lambda_n u}{u}\,du$$

$$= \lim_{\lambda_n\to\infty} h \int_0^{\delta} \left\{ f(x+u) - f(x-u) \right\} \frac{\cos\lambda_n u}{u}\,du$$

for any positive δ . Hence the result.

If the above limit is zero then the sum of the conjugate series is $f^*(x)$, and if there is a finite number of exponents

$$\pm\mu_1, \pm\mu_2, \ldots\ldots \pm\mu_j$$

in the interval (-1,1), the conditions for convergence will not be effected, for the contribution to the conjugate series would only be a t.p, and if we write this t.p as h(x), and

$$f(x) \;=\; h(x) \;+\; \phi(x)$$

as in Theorem (8.3), then by the same theorem the sum of the conjugate series is g(x).

(8.5) THEOREM.

(i) If $f \in U$, is even with exponents

$$0 = \lambda_0, \pm\lambda_1, \pm\lambda_2, \ldots\ldots\ldots\ldots$$

where $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n \longrightarrow \infty$.

(ii) f satisfies D for x = 0 (cf. Theorem (0), p.127).

(iii) The Fourier series of f converges to f(0) at x = 0.

Then the conjugate series of f converges to g(x) at a point where g(x) exists and f(x) satisfies $\bar{D}$. If the condition $\bar{D}$ is satisfied for all values of x, then the conjugate series converges almost everywhere to g(x).

Condition $\bar{D}$. f(x) satisfies $\bar{D}$ at x, if for some $\delta > 0$

$$\int_0^\delta \{f(x+u) - f(x-u)\} \frac{\cos \lambda u}{u} du \to 0 \quad \text{as } \lambda \to \infty.$$

Proof. As in Theorem (N) let

$$\Sigma_\lambda(x) = c_0 + 2 \sum_{\lambda_r \le \lambda} \left(1 - \frac{\lambda_r}{\lambda}\right) c_r \cos \lambda_r x \qquad (\lambda_r > 0).$$

Then
$$\lambda \Sigma_\lambda(0) = \lambda c_0 + 2 \sum_{0 < \lambda_r \le \lambda} (\lambda - \lambda_r) c_r$$
$$= \frac{2}{\pi} \int_0^\infty f(u) \frac{1 - \cos \lambda u}{u^2} du \qquad \text{(by Theorem (N))}$$
$$= p(\lambda), \text{ say.}$$

Now for $\lambda > 0$, and $\lambda_n \le \lambda \le \lambda_{n+1}$

$$p(\lambda) = \lambda c_0 + 2 \sum_{r=1}^{n} c_r (\lambda - \lambda_r) = \frac{2}{\pi} \int_0^\infty f(u) \frac{1 - \cos \lambda u}{u^2} du \quad \ldots\ldots (8.5.1)$$

Let x be such that x > 0, and g(x) exists, then

$$\int_0^{\lambda_m} p(\lambda) \sin \lambda x \, d\lambda = c_0 \int_0^{\lambda_1} \lambda \sin \lambda x \, d\lambda + \sum_{n=1}^{m-1} \int_{\lambda_n}^{\lambda_{n+1}} \{c_0 \lambda + 2 \sum_{r=1}^{n} c_r (\lambda - \lambda_r)\} \sin \lambda x \, d\lambda$$
$$= c_0 \int_0^{\lambda_m} \lambda \sin \lambda x \, d\lambda + 2 \sum_{r=1}^{m} c_r \int_{\lambda_r}^{\lambda_m} (\lambda - \lambda_r) \sin \lambda x \, d\lambda$$
$$= c_0 \left(-\rho \frac{\cos \rho x}{x} + \frac{\sin \rho x}{x^2}\right) + 2 \sum_{r=1}^{m} c_r \left\{ \frac{\sin \rho x}{x^2} + (\lambda_r - \rho) \frac{\cos \rho x}{x} - \frac{\sin \lambda_r x}{x^2} \right\}$$
$$(\rho = \lambda_m)$$

$$= -\frac{2}{x^2}\sum_1^m c_r \sin\lambda_r x - \frac{\cos\rho x}{x}\left\{\rho c_0 + 2\sum_1^m (\rho-\lambda_r)c_r\right\}$$
$$+ \frac{\sin\rho x}{x^2}\left\{c_0 + 2\sum_{r=1}^m c_r\right\}$$

$$= \frac{1}{x^2}\bar{S}_m(x) - \frac{\cos\rho x}{x}\phi(\rho) + \frac{\sin\rho x}{x^2}S_m(0)$$

$$= \frac{2}{\pi}\int_0^\infty \frac{f(u)}{u^2}\,du \int_0^\rho \sin\lambda x\,(1-\cos\lambda u)\,d\lambda$$

(by (8.5.1)).

Hence $\frac{1}{x^2}\bar{S}_m(x) + \frac{\sin\rho x}{x^2}S_m(0)$

$$= \frac{2}{\pi}\int_0^\infty \frac{f(u)}{u^2}\left\{\int_0^\rho \sin\lambda x\,(1-\cos\lambda u)\,d\lambda\right\} + \frac{\cos\rho x}{x}\frac{2}{\pi}\int_0^\infty \frac{f(u)}{u^2}(1-\cos\rho u)\,du$$

(again by (8.5.1)),

or $\bar{S}_m(x) + \sin\rho x\, S_m(0)$

$$= \frac{2x^2}{\pi}\int_0^\infty \frac{f(u)}{u^2}\int_0^\rho \sin\lambda x\,(1-\cos\lambda u)\,d\lambda + \frac{2x}{\pi}\cos\rho x\int_0^\infty f(u)\,\frac{1-\cos\rho u}{u^2}\,du$$

$$= \frac{2}{\pi}\int_0^\infty \frac{f(u)}{u^2}\left\{x - x\cos\rho x\cos\rho u + \frac{x^2}{2}\left(\frac{1}{u-x} - \frac{1}{u+x}\right) + \right.$$
$$\left. + \frac{x^2}{2}\frac{\cos\rho(u+x)}{u+x} - \frac{x^2}{2}\frac{\cos\rho(u-x)}{u-x}\right\}du$$

$$= \frac{2}{\pi}\int_0^\infty \frac{f(u)}{u^2}\left\{\frac{u^2x}{u^2-x^2} - x\cos\rho x\cos\rho u + \frac{x^2}{2}\frac{\cos\rho(u+x)}{u+x} - \right.$$
$$\left. - \frac{x^2}{2}\frac{\cos\rho(u-x)}{u-x}\right\}du$$

$$= \frac{2x}{\pi} P\int_0^\infty \frac{f(u)}{u^2-x^2}\,du - \frac{2x\cos\rho x}{\pi}\int_\delta^\infty \frac{f(u)\cos\rho u}{u^2}\,du +$$

$$+ \frac{x^2}{\pi}\int_\delta^\infty \frac{f(u)}{u^2}\cdot\frac{\cos\rho(u+x)}{u+x}\,du - \frac{x^2}{\pi}\left[\int_\delta^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^\infty\right]\frac{f(u)}{u^2}\cdot\frac{\cos\rho(u-x)}{u-x}\,du$$

$$+ \int_0^\delta \left\{\frac{x^2}{\pi}\,\frac{f(u)}{u^2}\cdot\frac{\cos\rho(u+x)}{u+x} - \frac{x^2}{\pi}\frac{f(u)}{u^2}\cdot\frac{\cos\rho(u-x)}{u-x} - \frac{2x\cos\rho x}{\pi}\,\frac{f(u)}{u^2}\cos\rho u\right\}du$$

for some $\delta > 0$, such that $0 < \delta < x - \delta$.

Now as f(x) is even

$$\frac{2x}{\pi} P\int_0^\infty \frac{f(u)}{u^2-x^2} = g(x)$$

$\int_\delta^\infty \frac{f(u)}{u^2}\,\frac{\cos\rho(u+x)}{u+x}\,du \longrightarrow 0$ as $\rho\to\infty$, (by Riemann-Lebesgue Theorem),

and similarly

$$\int_\delta^\infty \frac{f(u)}{u^2}\cos\rho u\,du \longrightarrow 0 \text{ as } \rho\longrightarrow\infty.$$

Again $\left(\int_\delta^{x-\delta} \text{ and } \int_{x+\delta}^\infty\right)\frac{f(u)}{u^2}\,\frac{\cos\rho(u-x)}{u-x}\,du \to 0$ as $\rho\longrightarrow\infty$,

$$x^2 P\int_{x-\delta}^{x+\delta}\frac{f(u)}{u^2}\cdot\frac{\cos\rho(u-x)}{u-x}\,du = P\int_{x-\delta}^{x+\delta} f(u)\,\frac{x^2\cos\rho(u-x)}{u^2(u-x)}\,du$$

$$= P\int_{x-\delta}^{x+\delta} f(u)\left\{\frac{1}{u-x} - \frac{1}{u} - \frac{x}{u^2}\right\}\cos\rho(u-x)\,du$$

$$= O(1) + P\int_{x-\delta}^{x+\delta} f(u)\,\frac{\cos\rho(u-x)}{u-x}\,du,$$

and

$$\frac{x}{\pi}\int_0^{\delta}\left\{\frac{x\cos\rho(u+x)}{u+x}-\frac{x\cos\rho(u-x)}{u-x}-2\cos\rho x\cos\rho u\right\}\frac{f(u)}{u^2}\,du$$

$$=\frac{x}{\pi}\int_0^{\delta}\frac{f(u)}{u^2}\left\{\frac{x(u-x)\cos\rho(u+x)-(u+x)x\cos\rho(u-x)-2(u^2-x^2)\cos\rho x\cos\rho u}{u^2-x^2}\right\}du$$

$$=\frac{x}{\pi}\int_0^{\delta}\frac{f(u)}{u^2(u^2-x^2)}\left\{ux\left(\cos\rho(u+x)-\cos\rho(u-x)\right)-\right.$$

$$\left.-x^2\left(\cos\rho(u+x)+\cos\rho(u-x)-2(u^2-x^2)\cos\rho x\cos\rho u\right\}du\right.$$

$$=\frac{2x}{\pi}\int_0^{\delta}\frac{f(u)}{u^2(u^2-x^2)}\left[-ux\sin\rho u\sin\rho x-x^2\cos\rho u\cos\rho x-\right.$$

$$\left.-(u^2-x^2)\cos\rho x\cos\rho u\right]du$$

$$=-\frac{2x}{\pi}\int_0^{\delta}\frac{u\,f(u)}{u^2(u^2-x^2)}\left[x\sin\rho u\sin\rho x+u\cos\rho x\cos\rho u\right]du$$

$$=-\frac{2x^2\sin\rho x}{\pi}\int_0^{\delta}\frac{f(u)}{u^2-x^2}\frac{\sin\rho u}{u}\,du-\frac{2x\cos\rho x}{\pi}\int_0^{\delta}\frac{f(u)}{u^2-x^2}\cos\rho u\,du.$$

Again $\displaystyle\int_0^{\delta}\frac{f(u)}{u^2-x^2}\cos\rho u\,du\longrightarrow 0$ as $\rho\longrightarrow\infty$, $\quad(x>\delta>0)$

and $$-x^2\int_0^{\delta}\frac{f(u)}{u^2-x^2}\,\frac{\sin\rho u}{u}\,du$$

$$= -\tfrac{1}{2}\int_0^{\delta}\frac{f(u)}{u-x}\sin\rho u\,du - \tfrac{1}{2}\int_0^{\delta}\frac{f(u)}{u+x}\sin\rho u\,du + \int_0^{\delta} f(u)\,\frac{\sin\rho u}{u}\,du.$$

The first two integrals on the right tend to zero as $\rho \to \infty$, by Riemann- Lebesgue Theorem.

Hence combining the above results we get

$$\bar{S}_m(x) + \sin\rho x\; S_m(0) = o(1) - \frac{1}{\pi}\int_{x-\delta}^{x+\delta} f(u)\,\frac{\cos\rho(u-x)}{u-x}\,du$$

$$+ g(x) + \frac{2}{\pi}\sin\rho x\int_0^{\delta} f(u)\,\frac{\sin\rho u}{u}\,du$$

$$\bar{S}_m(x) = o(1) + \int_0^{\delta}\{f(x+u) - f(x-u)\}\,\frac{\cos\rho u}{u}\,du -$$

$$- \sin\rho x\left\{S_m(0) - \frac{2}{\pi}\int_0^{\delta} f(u)\,\frac{\sin\rho u}{u}\,du\right\} + g(x).$$

Where $$\bar{S}_m(x) = -2\sum_1^m c_r \sin\lambda_r x,$$

and $$S_m(0) = c_0 + 2\sum_1^m c_r.$$

Hence the required result.

REFERENCES

[1]. Titchmarsh, E. C. Reciprocal formulae involving series and integrals. Math. Z., 25 (1926), 321 - 47.

[2]. Titchmarsh, E. C. A series inversion formula. Proc. Lond. Math. Soc., (2), 26 (1927), 1 - 11.

[3]. Cossar, J. Hilbert transforms and almost periodic functions. Proc Camb. Phil. Soc.,(4), 56 (1960), 354 - 66.

[4]. Boas, R. P. Some theorems on Fourier transforms and conjugate trigonometric integrals. Trans. American Math. Soc., 40 (1936), 287 - 308.

[5]. Goldberg, R. R. An integral transform related to the Hilbert transform. J. London Math. Soc., 35 (1960), 200 - 4.

[6]. Heywood, P. On a transform discussed by Goldberg. J. London Math. Soc., 38 (1963), 162 - 8.

[7]. Cossar, J. Research lectures on almost periodic functions delivered during the spring term of 1963.

[8]. Titchmarsh, E. C. Introduction to the theory of Fourier integrals (Oxford, 1937).

[9]. Zygmund, A. Trigonometric series, Vol. I (Cambridge, 1959).

A P P E N D I X

f(x)	F(x)	g(x)
(1). $k(x) = \sqrt{\frac{2}{\pi}}\,\frac{\sin x}{x}$, $K(x) = \begin{matrix}1 & (|x| \le 1)\\ 0 & (|x| > 1)\end{matrix}\Big\}$		$-\sqrt{\frac{2}{\pi}}\,\frac{1-\cos x}{x}$
(2). $\delta(x) = \sqrt{\frac{2}{\pi}}\,\frac{1-\cos x}{x^2}$, $\Delta(x) = \begin{matrix}1-|x| & (|x|\le 1)\\ 0 & (|x|>1)\end{matrix}\Big\}$		$\sqrt{\frac{2}{\pi}}\left(\frac{\sin x}{x^2} - \frac{1}{x}\right)$
(3). $\sigma(x) = \sqrt{\frac{2}{\pi}}\int_1^\infty \frac{\sin xu}{u}\,du$, $\Sigma(x) = \begin{matrix}0 & (|x| \le 1)\\ \frac{i}{x} & (|x| > 1)\end{matrix}\Big\}$		$\sqrt{\frac{2}{\pi}}\int_1^\infty \frac{\cos xu}{u}\,du$

(i). The Fourier transform of g(x) is, by (0.3), $-i\,\mathrm{sgn}x\,F(x)$. Hence, for example, the Fourier transform of

$$\sqrt{\frac{2}{\pi}}\,\frac{1-\cos x}{x}$$

is

$$\begin{matrix} i\,\mathrm{sgn}x & (|x| \le 1)\\ 0 & (|x| > 1).\end{matrix}\Big\}$$

(ii). The Goldberg transform of the first two functions are the same as their derivatives, since they satisfy the conditions of Theorem (2.4). The third function, on the other hand, satisfies Theorem(D) and as such its Goldberg and Hilbert transforms are equal.